A quick note on advection of localization

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/home/ross/DataAssim/Notes/AdvectionLocalization.lyx

Without advection the 4D Schur product is

$$\underline{\mathbf{P}}\circ\underline{\mathbf{L}}=\begin{pmatrix}\mathbf{P}_{11} & \mathbf{P}_{12} & \cdots & \mathbf{P}_{1L}\\ \mathbf{P}_{21} & \mathbf{P}_{22} & & \\ \vdots & & & \\ \mathbf{P}_{L1} & & & \mathbf{P}_{LL}\end{pmatrix}\circ\begin{pmatrix}\mathbf{L} & \mathbf{L} & \cdots & \mathbf{L}\\ \mathbf{L} & \mathbf{L} & & \\ \vdots & & & \\ \mathbf{L} & & & \mathbf{L}\end{pmatrix},$$

where $\mathbf{P}_{ij}$ is the ensemble's 3D sample error covariance matrix between time steps i and j, $\mathbf{L}$ is the 3D localization correlation matrix, and the underlines indicate a 4D operator. Note that

$$\begin{pmatrix}\mathbf{L} & \mathbf{L} & \cdots & \mathbf{L}\\ \mathbf{L} & \mathbf{L} & & \\ \vdots & & & \\ \mathbf{L} & & & \mathbf{L}\end{pmatrix}=\begin{pmatrix}\mathbf{I}\\ \mathbf{I}\\ \vdots\\ \mathbf{I}\end{pmatrix}\mathbf{L}\begin{pmatrix}\mathbf{I} & \mathbf{I} & \cdots & \mathbf{I}\end{pmatrix}=\begin{pmatrix}\mathbf{L}^{1/2}\\ \mathbf{L}^{1/2}\\ \vdots\\ \mathbf{L}^{1/2}\end{pmatrix}\begin{pmatrix}\mathbf{L}^{\mathrm{T}/2} & \mathbf{L}^{\mathrm{T}/2} & \cdots & \mathbf{L}^{\mathrm{T}/2}\end{pmatrix}.$$

The square-root of this is

$$\underline{\mathbf{L}}^{1/2}=\begin{pmatrix}\mathbf{L}^{1/2}\\ \mathbf{L}^{1/2}\\ \vdots\\ \mathbf{L}^{1/2}\end{pmatrix}.$$

Recall that the square-root of the localization matrix can be interpreted as a matrix of quasi ensemble members whose covariance is the localization. This allows us to introduce advection into the problem by simply replacing the above with

$$\underline{\mathbf{L}}^{1/2}=\begin{pmatrix}\mathbf{L}^{1/2}\\ \mathbf{M}\mathbf{L}^{1/2}\\ \vdots\\ \mathbf{M}^{T-1}\mathbf{L}^{1/2}\end{pmatrix},$$

where $\mathbf{M}$ is the advection operator (for one time step).