Working paper on possible strategies for background error covariance modelling for convective-scale data assimilation

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1. Introduction

A new approach to data assimilation is expected to be required for meteorological systems that involve convective-scale motion, as opposed to synoptic and global-scale motion which is currently considered. For synoptic and global scales the atmosphere has certain physical properties (viz. geostrophic and hydrostatic balance), which are exploited for the purpose of modelling the background error covariances. These properties may break down at convective- scales which leads us to re-examine the data assimilation problem for such motions.

Background error covariance statistics describe the probability density function of (Gaussian) background errors of the variables that are usually represented in a model forecast (the so-called 'model variables'). These statistics are a very important part of data assimilation as they specify how a background state (otherwise known as an 'a-priori' or a 'first guess' state) is allowed to be modified by observations. The success of a data assimilation system can depend strongly on how the background error covariances are specified. For instance they can make the difference between a data assimilation system that produces realistic and sufficiently smooth analyses which are appropriate to the system, and one that does not.

Background error covariances are modelled using a technique called control variable transforms (CVTs). This technique attempts to re-express the cost function from one in terms of model variables to another in terms of new variables (called 'control variables') whose background errors are uncorrelated (for a review, see e.g. Bannister 2008). This is difficult to do exactly as the variables that are uncorrelated are usually unknown, but schemes can be proposed that *assume* certain carefully chosen variables are uncorrelated. Such assumptions form the basis of a model of the background error covariances. Such a model gives rise to so-called implied background error covariances of model variables which should be as close as possible to the actual background error covariances (a subset of which can usually be estimated explicitly to help evaluate the implied covariances).

Currently, the Met Office's control variables are streamfunction ($\delta\psi$), unbalanced velocity potential ($\delta\chi^{\mathrm{u}}$), geostrophically unbalanced pressure (δp^{A}) and a relative humidity variable ($\delta\mu$), where the δ preceding each variable denotes an error (or perturbation) in each quantity. It is assumed that background errors between the control variables are uncorrelated. This choice of variables is most appropriate at larger-scales, where it is possible to assume that geostrophic and hydrostatic balances are important. The scheme works in the data assimilation by recovering the model variables from these control variables using a CVT. The inverse CVT is also needed to derive control variables from model variables, which is an essential off-line step needed to determine the spatial statistics of the control variables. For instance, p^{A} is found from $\delta\psi$ and total pressure error (δp) using a balance relation - specifically the linear balance equation (LBE) (roughly equivalent to geostrophic balance) as follows

$$\delta p^{\mathrm{A}} = \delta p - \mathbf{L}\delta\psi, \tag{1.1}$$

$$\text{where} \qquad \mathbf{L}\delta\psi = \nabla_{\mathrm{h}}^{-2}\left[\nabla_{\mathrm{h}} \cdot (f\rho\nabla_{\mathrm{h}}\delta\psi)\right], \tag{1.2}$$

is the diagnosed balanced pressure. Here f is the Coriolis parameter, ρ is reference state density, ∇_{h} is the horizontal gradient operator and **L** is the linear balance operator defined in (1.2). This scheme is useful at scales where the LBE holds by assuming that all of the $\delta\psi$ field is balanced (i.e. there is no unbalanced streamfunction). The scheme works even in the tropics where the balance diminishes in a specified way ($f \rightarrow 0$ as the latitude $\rightarrow 0$). Hence at the equator, all pressure errors are unbalanced, $\delta p^{\mathrm{A}} = \delta p$. In a similar way temperature (δT) can be diagnosed from δp using an operator that can be derived from hydrostatic balance - see e.g. Bannister (2008)

$$\delta T = \frac{g}{c_p}\left(\frac{\partial \Pi}{\partial z}\right)^{-1}\left\{\Pi\left(\frac{\partial \Pi}{\partial z}\right)^{-1}\frac{\partial}{\partial z} - 1\right\}\frac{\kappa\Pi}{p}\delta p. \tag{1.3}$$

Here g is the acceleration due to gravity, c_p is the specific heat capacity at constant pressure, z is height, κ is the ratio of specific heats, p is the reference state pressure and Π is the reference state exner pressure. (N.B. there is also a moisture contribution to δT, which has been neglected in the above.)

At convective scales, the importance of linear and hydrostatic balances are known to diminish (e.g. Vetra-Carvalho et al., 2010; Bannister et al., 2011). In the case of extremely small scales for instance, where geostrophic balance does not hold at all, we would expect all pressure errors to be unbalanced, $\delta p^{\mathrm{A}} \approx \delta p$. Equation (1.1), however, is inconsistent with this expectation given that $\mathbf{L}\delta\psi$ may have a substantial value. Unfortunately, unlike the midlatitude vs. tropical scenarios (where linear balance respectively does and doesn't apply), there is nothing in (1.1) to distinguish between large and small scales (this can be seen easily in the case when f and ρ are constants where (1.2) becomes $\mathbf{L}\delta\psi = f\rho\delta\psi$). A similar argument holds for hydrostatic balance, where (1.3) would not be the correct relation to use to diagnose δT, although the extent to which (1.3) is inappropriate in not yet clear (Bannister et al., 2011).

There are two problems here. The first is to decide what to do specifically for convective-scale flows where (1.1)-(1.3) may be inappropriate (called the 'convective-scale' problem), and the second is to decide how to treat the convective-scale flows simultaneously with the large-scale part of the problem where (1.1)-(1.3) remain appropriate (called the 'multi-scale' problem). The following possible strategies may apply for the convective-scale problem and each recognises the diminished roles of geostrophic and hydrostatic balances at convective scales.

I. Do not decompose variables into balanced and unbalanced variables, i.e. treat 'model variables' themselves as control variables.
II. Use the same control variables that are used currently, but turn-off balance relations.
III. Introduce extra control variables that reflect the unbalanced nature of the atmosphere at convective-scales.
IV. Propose alternative diagnostic relationships that do hold at convective-scales, to replace or complement the traditional balance relations.
V. Introduce a set of purely statistical relationships (instead of diagnostic relationships) in the control variable transforms, which are valid at convective-scales.
VI. Define forecast errors in the representation of the normal modes of the forecast model (linearized about the background). (This strategy has been taken by Ruth Petrie for her PhD thesis - see refs.).

The following possible strategies may apply for the multi-scale problem.

i. Look for universally relevant variables, which are always approximately uncorrelated where the associated 'balance' relationships (yet to be determined) inherently adjust to the dynamic regime (i.e. midlatitudes, tropics, large-scale or convective-scale).

ii. Use regime-dependent variables. This means two sets of variables, each representing a different scale regime (i.e. one set that applies at large-scales and one set that applies at convective-scales). A cross-over length scale must be applied.

iii. Run an assimilation where only the convective-scales are adjusted by the main control variables. Large-scale information (from a conventional assimilation performed beforehand) may be introduced by means of a separate constraint in the cost function (the so-called J_K term as in Fischer et al., 2005).

Clearly, these are complicated issues and there will not be time to develop and compare all in full. In this working paper the current scheme is outlined and then two significant modifications are made using III, IV and ii above. It is assumed that the best approach is to solve the synoptic/convective-scales together in the same data assimilation problem and so option iii is not considered here.

2. The current transforms

In the current scheme, the model variables are δu, δv, δw, $\delta\theta$, $\delta\rho$, δp and δq_{T} and the control variables are $\delta\psi$, $\delta\chi^{\mathrm{u}}$, δp^{A}, $\delta\mu$. There are seven model variables, but only four control variables. When converting from control-to-model variables, the three missing fields come from imposing hydrostatic balance, the ideal gas law and incompressibility.

2.1 The current Tp-transform

Input fields	Output fields
δu, δv, $\delta\theta$, δp and δq_{T}	$\delta\psi$, $\delta\chi^{\mathrm{u}}$, δp^{A}, $\delta\mu$

The Tp-transform is shorthand for the set of transformations that go from model to control variables that are thought to be uncorrelated. The current Tp-transform is the following (see Met Office, 2010). In the following, many of the steps are intermediate. The steps that result in the control variable fields are marked with an asterisk. A shorthand form of each equation (assuming a matrix/vector notation) is given with each step with equation numbers appended with an "a".

Tp.1. Calculate the virtual potential temperature

$$\delta\theta^{\mathrm{v}} = [1 + (\varepsilon^{-1} - 1)q]\,\delta\theta + \theta(\varepsilon^{-1} - 1)\,\delta q_{\mathrm{T}}, \tag{2.1}$$

$$\delta\theta^{\mathrm{v}} = \Theta^{\theta}\delta\theta + \Theta^{q}\delta q_{\mathrm{T}}, \tag{2.1a}$$

where ε is the ratio of the molecular weight of water to the molecular weight of dry air.

Tp.2. Calculate the hydrostatic exner pressure by integrating the hydrostatic equation

$$\delta\Pi^{\mathrm{H}}_{k+½} = \delta\Pi^{\mathrm{H}}_{k-½} + \frac{g\,(z_{k+½} - z_{k+½})}{c_p\theta^{\mathrm{v}2}}\delta\theta^{\mathrm{v}}, \tag{2.2}$$

$$\delta\Pi^{\mathrm{H}} = \mathbf{P}_0^{-1}\delta\theta^{\mathrm{v}}, \tag{2.2a}$$

where Π^{H} is the hydrostatic exner pressure, g is the acceleration of gravity, c_p is the specific heat capacity at constant pressure and z is the model level height.

Tp.3. Calculate the moisture control variable

$$\delta\mu = a\left(\frac{1}{q_{\mathrm{s}}}\delta q_{\mathrm{T}} - h_1\frac{q\Pi}{q_{\mathrm{s}}}\frac{d\,\ln e_{\mathrm{s}}}{dT}\delta\theta - h_2\frac{q}{pq_{\mathrm{s}}}\left[\kappa\Pi\theta\frac{d\,\ln e_{\mathrm{s}}}{dT} - 1\right]\delta p\right), \tag{2.3*}$$

$$\delta\mu = \mathbf{M}^{q}\delta q_{\mathrm{T}} + \mathbf{M}^{\theta}\delta\theta + \mathbf{M}^{p}\delta p, \tag{2.3a}$$

where a is a normalization constant, q_{s} in the saturated humidity mixing ratio, e_{s} is the saturated vapour pressure of water, κ is the ratio of specific heats, and h_1 and h_2 are known correlation coefficients. To introduce a language, $\delta\mu$ may be referred to as an 'unbalanced' variable because the 'balanced' contributions from $\delta\theta$ and δp have been removed. The contributions to $\delta\mu$ from $\delta\theta$ and δp are termed "balanced" because they are associated with (or in 'balance' with) $\delta\theta$ and δp. The words "balanced" and "unbalanced" have different meanings to those used with respect to balance relations like hydrostatic or geostrophic balance.

Tp.4. Calculate the streamfunction

$$\delta\psi \;=\; \nabla_{\mathrm{h}}^{-2}\left\{\mathbf{k}\cdot\left[\nabla\times\begin{pmatrix}\delta u\\ \delta v\\ 0\end{pmatrix}\right]\right\}, \tag{2.4*}$$

$$\delta\psi \;=\; \mathbf{Y}\delta\mathbf{u}_{w=0}, \tag{2.4a}$$

where $\mathbf{k}$ is the vertical unit vector, ∇_{h} comprises the horizontal components of the gradient vector, ∇ comprises all three components of the gradient vector and $\delta\mathbf{u}_{w=0}$ is a 3-D wind vector with zero vertical component.

Tp.5. Calculate the velocity potential

$$\delta\chi \;=\; \nabla_{\mathrm{h}}^{-2}\left\{\nabla\cdot\begin{pmatrix}\delta u\\ \delta v\\ 0\end{pmatrix}\right\}. \tag{2.5}$$

$$\delta\chi \;=\; \mathbf{C}\delta\mathbf{u}_{w=0}, \tag{2.5a}$$

Tp.6. Calculate the balanced component of the velocity potential

$$\delta\chi^{\mathrm{b}} \;=\; \mathbf{B}_{\mathrm{v}}^{\chi\psi}\mathbf{B}_{\mathrm{v}}^{\psi\psi^{-1}}\delta\psi, \tag{2.6}$$

$$\;=\; \mathbf{X}\delta\psi, \tag{2.6a}$$

where $\mathbf{B}_{\mathrm{v}}^{\chi\psi}$ is the vertical error covariance between $\delta\psi$ and $\delta\chi$, and $\mathbf{B}_{\mathrm{v}}^{\psi\psi}$ is the vertical error covariance between $\delta\psi$ and itself.

Tp.7. Calculate the unbalanced component of the velocity potential

$$\delta\chi^{\mathrm{u}} \;=\; \delta\chi - \delta\chi^{\mathrm{b}}, \tag{2.7* / 2.7a}$$

Tp.8. Calculate the level-by-level geostrophically balanced pressure

$$\delta p^{\mathrm{G}} \;=\; \nabla_{\mathrm{h}}^{-2}\left\{\nabla_{\mathrm{h}}\cdot\left[f\rho\nabla_{\mathrm{h}}\delta\psi\right]\right\}, \tag{2.8}$$

$$\delta p^{\mathrm{G}} \;=\; \mathbf{L}\delta\psi. \tag{2.8a}$$

Tp.9. Calculate the vertically regressed geostrophically balanced pressure

$$\delta p^{\mathrm{F}} \;=\; \mathbf{B}_{\mathrm{v}}^{p^{\mathrm{F}}p^{\mathrm{G}}}\mathbf{B}_{\mathrm{v}}^{p^{\mathrm{G}}p^{\mathrm{G}^{-1}}}\delta p^{\mathrm{G}}, \tag{2.9}$$

$$\delta p^{\mathrm{F}} \;=\; \mathbf{G}\delta p^{\mathrm{G}}. \tag{2.9a}$$

The geostrophically balanced pressure calculated in step 7 has to be regressed vertically to ensure vertical consistency between levels. Performing step 7 alone, which is performed level-by-level, is problematic as it does not ensure continuity between neighbouring levels.

Tp.10. Calculate the hydrostatic pressure

$$\delta p^{\mathrm{H}} \;=\; \frac{p^{\mathrm{H}}}{\kappa\Pi^{\mathrm{H}}}\delta\Pi^{\mathrm{H}}, \tag{2.10}$$

$$\delta p^{\mathrm{H}} \;=\; \mathbf{P}_1^{-1}\delta\Pi^{\mathrm{H}}. \tag{2.10a}$$

Combining (2.10a) with (2.2a) gives the shorthand

$$\delta p^{\mathrm{H}} = \mathbf{P}^{-1}\delta\theta^{\mathrm{v}}, \qquad (2.10b)$$

where $\mathbf{P}^{-1} = \mathbf{P}_1^{-1}\mathbf{P}_0^{-1}$.

Tp.11. Calculate the geostrophically unbalanced pressure (ageostrophic pressure)

$$\delta p^{\mathrm{A}} = \delta p^{\mathrm{H}} - \delta p^{\mathrm{F}}. \qquad (2.11^*/2.11a)$$

Note that in this Tp-transform the model variables δw and $\delta\rho$ are not used and δp is used only in the calculation of $\delta\mu$ and to define the lower boundary condition in the calculation of $\delta\Pi^{\mathrm{H}}$ (not shown here). The Tp-transform is used only when VAR outer loops are invoked and for the (off-line) calibration procedure.

2.2 The current Up-transform

Input fields	Output fields
$\delta\psi$, $\delta\chi^{\mathrm{u}}$, δp^{A}, $\delta\mu$	δu, δv, δw, $\delta\theta$, $\delta\rho$, δp and δq_{T}

The Up-transform is shorthand for the set of transformations that go from control to model variables. The current Tp-transform is the following (see Met Office, 2010). The Up-transform is used at every VAR iteration and so efficiency is a very important consideration for its implementation. The steps that result in the model fields are marked with an asterisk. The same shorthand form of each equation as used alongside the Tp-transform (assuming a matrix/vector notation) is given with each step with equation numbers appended with an "a".

Up.1. Calculate the balanced velocity potential (as Tp.6)

$$\delta\chi^{\mathrm{b}} = \mathbf{B}_{\mathrm{v}}^{\chi\psi}\mathbf{B}_{\mathrm{v}}^{\psi\psi^{-1}}\delta\psi, \qquad (2.12)$$

$$= \mathbf{X}\delta\psi. \qquad (2.12a)$$

Up.2. Calculate the velocity potential (as Tp.7)

$$\delta\chi = \delta\chi^{\mathrm{u}} + \delta\chi^{\mathrm{b}}. \qquad (2.13/2.13a)$$

Up.3. Calculate the horizontal wind components (as Tp.4 and Tp.5)

$$\begin{pmatrix}\delta u\\ \delta v\end{pmatrix} = \nabla_{\mathrm{h}}\delta\chi + \mathbf{k}\times\nabla\delta\psi, \qquad (2.14^*)$$

$$\delta\mathbf{u}_2 = \mathbf{C}^{-1}\delta\chi + \mathbf{Y}^{-1}\delta\psi, \qquad (2.14a)$$

where $\delta\mathbf{u}_2$ is a 2-D (horizontal) wind vector.

Up.4. Calculate the level-by-level geostrophically balanced pressure (as Tp.8)

$$\delta p^{\mathrm{G}} = \nabla_{\mathrm{h}}^{-2}\{\nabla_{\mathrm{h}}\cdot[f\rho\nabla_{\mathrm{h}}\delta\psi]\}, \qquad (2.15)$$

$$\delta p^{\mathrm{G}} = \mathbf{L}\delta\psi. \qquad (2.15a)$$

Up.5. Calculate the vertically regressed geostrophically balanced pressure (as Tp.9)

$$\delta p^{\mathrm{F}} = \mathbf{B}_{\mathrm{v}}^{p^{\mathrm{F}}p^{\mathrm{G}}}\mathbf{B}_{\mathrm{v}}^{p^{\mathrm{G}}p^{\mathrm{G}^{-1}}}\delta p^{\mathrm{G}}, \qquad (2.16)$$

$$\delta p^{\mathrm{F}} = \mathbf{G}\delta p^{\mathrm{G}}. \tag{2.16a}$$

Up.6. Calculate the hydrostatically balanced pressure (as Tp.11)

$$\delta p^{\mathrm{H}} = \delta p^{\mathrm{A}} + \delta p^{\mathrm{F}}. \qquad (2.17/2.17\mathrm{a})$$

Up.7. Calculate the virtual potential temperature (as Tp.2/Tp.10)

$$\delta\theta^{\mathrm{v}} = \frac{\kappa g}{c_p}\left(\frac{\partial\Pi}{\partial z}\right)^{-2}\frac{\partial}{\partial z}\left(\frac{\Pi^{\mathrm{H}}\delta p^{\mathrm{H}}}{p^{\mathrm{H}}}\right), \tag{2.18}$$

$$\delta\theta^{\mathrm{v}} = \mathbf{P}\delta p^{\mathrm{H}}. \tag{2.18a}$$

Up.8. Calculate the pressure

$$\delta p = \delta p^{\mathrm{H}}, \tag{2.19*}$$

(as all pressure is assumed to be hydrostatic).

Up.9. Calculate together the potential temperature and the total specific humidity (as Tp.1 and Tp.3)

$$\delta q_{\mathrm{T}} = \left\{ah_1\frac{q\Pi}{q_{\mathrm{s}}}\frac{d\,\ln e_{\mathrm{s}}}{dT}\delta\theta^{\mathrm{v}} + [1 + (\varepsilon^{-1} - 1)q]\,\delta\mu + [1 + (\varepsilon^{-1} - 1)q]\,ah_2\frac{q}{pq_{\mathrm{s}}}\left[\kappa\Pi\theta\frac{d\,\ln e_{\mathrm{s}}}{dT} - 1\right]\delta p\right\} / \left\{ah_1\frac{q\Pi}{q_{\mathrm{s}}}\frac{d\,\ln e_{\mathrm{s}}}{dT}\theta(\varepsilon^{-1} - 1) + [1 + (\varepsilon^{-1} - 1)q]\,\frac{a}{q_{\mathrm{s}}}\right\}, \tag{2.20*}$$

$$\delta q_{\mathrm{T}} = \mathbf{A}^{-1}\left\{\Theta^{\theta}\delta\mu - \mathbf{M}^{\theta}\delta\theta^{\mathrm{v}} - \Theta^{\theta}\mathbf{M}^{p}\delta p\right\}, \tag{2.20a}$$

$$\delta\theta = \left\{\frac{a}{q_{\mathrm{s}}}\delta\theta^{\mathrm{v}} - \theta(\varepsilon^{-1} - 1)\,\delta\mu - \theta(\varepsilon^{-1} - 1)\,ah_2\frac{q}{pq_{\mathrm{s}}}\left[\kappa\Pi\theta\frac{d\,\ln e_{\mathrm{s}}}{dT} - 1\right]\delta p\right\} / \left\{ah_1\frac{q\Pi}{q_{\mathrm{s}}}\frac{d\,\ln e_{\mathrm{s}}}{dT}\theta(\varepsilon^{-1} - 1) + [1 + (\varepsilon^{-1} - 1)q]\,\frac{a}{q_{\mathrm{s}}}\right\}, \tag{2.21*}$$

$$\delta\theta = \mathbf{A}^{-1}\left\{\Theta^{q}\delta\mu - \mathbf{M}^{q}\delta\theta^{\mathrm{v}} - \Theta^{q}\mathbf{M}^{p}\delta p\right\}, \tag{2.21a}$$

which form the solution of (2.1) and (2.3) solved simultaneously. In (2.20a) and (2.21a), $\mathbf{A}$ is

$$\mathbf{A} = \Theta^{\theta}\mathbf{M}^{q} - \Theta^{q}\mathbf{M}^{\theta}. \tag{2.22}$$

Up.10. Calculate the density

$$\delta\rho = \rho\left(\frac{1 - \kappa}{p}\delta p - \frac{1}{\theta^{\mathrm{v}}}\delta\theta^{\mathrm{v}}\right), \tag{2.23*}$$

$$\delta\rho = \mathbf{R}^{p}\delta p + \mathbf{R}^{\theta}\delta\theta^{\mathrm{v}}, \tag{2.23a}$$

which is the linearization of the equation of state, $p = \rho RT^{\mathrm{v}}$, where $T^{\mathrm{v}} = (p/p_{1000})^{\kappa}\,\theta^{\mathrm{v}}$.

Up.11. Calculate the vertical velocity component by solving either the incompressible equation

$$\frac{\partial\,(\delta w \rho_y)}{\partial z} + \frac{\partial\,(\delta \rho_y w)}{\partial z} = -\nabla_{\mathrm{h}} \cdot \left[\rho_y \begin{pmatrix} \delta u \\ \delta v \end{pmatrix}\right] - \nabla_{\mathrm{h}} \cdot \left[\delta\rho_y \begin{pmatrix} u \\ v \end{pmatrix}\right], \tag{2.24*}$$

$$\delta w \approx \mathbf{W}\delta\mathbf{u}_2, \tag{2.24a}$$

where ρ_y is the 'dry density', or by solving Richardson's equation (not shown). For simplicity, the matrix/vector version of (2.24) does not include the $\delta\rho$ contribution.

2.3 The implied background error covariance matrix for the current transforms

We now examine the implied background error covariance matrix for the current transforms. The expressions can become very complicated, even for this simple scheme and so we examine the implied covariances for the variables $\delta\psi$, $\delta\chi$, δp and $\delta\theta^{\mathrm{v}}$ only. Consequently, we need to involve the control variables $\delta\psi$, $\delta\chi^{\mathrm{u}}$ and δp^{A} only. First, a summary of the current Up-transform for this reduced variable set

$$\begin{pmatrix} \delta\psi \\ \delta\chi \\ \delta p \\ \delta\theta^{\mathrm{v}} \end{pmatrix} = \begin{pmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{X} & \mathbf{I} & \mathbf{0} \\ \mathbf{GL} & \mathbf{0} & \mathbf{I} \\ \mathbf{PGL} & \mathbf{0} & \mathbf{P} \end{pmatrix} \begin{pmatrix} \delta\psi \\ \delta\chi^{\mathrm{u}} \\ \delta p^{\mathrm{A}} \end{pmatrix}. \tag{2.25}$$

The implied covariances are

$$\mathbf{B}^{\mathrm{imp}} = \mathbf{U}_p \mathbf{B}_p \mathbf{U}_p^{\mathrm{T}}, \tag{2.26}$$

where $\mathbf{B}_p$ is the background error covariance matrix of the variables $\delta\psi$, $\delta\chi^{\mathrm{u}}$ and δp^{A} (which is block diagonal). This expands to

$$\mathbf{B}^{\mathrm{imp}} = \begin{pmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{X} & \mathbf{I} & \mathbf{0} \\ \mathbf{GL} & \mathbf{0} & \mathbf{I} \\ \mathbf{PGL} & \mathbf{0} & \mathbf{P} \end{pmatrix} \begin{pmatrix} \mathbf{B}_p^{\psi} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_p^{\chi^{\mathrm{u}}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{B}_p^{p^{\mathrm{A}}} \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{X}^{\mathrm{T}} & \mathbf{L}^{\mathrm{T}}\mathbf{G}^{\mathrm{T}} & \mathbf{L}^{\mathrm{T}}\mathbf{G}^{\mathrm{T}}\mathbf{P}^{\mathrm{T}} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{P}^{\mathrm{T}} \end{pmatrix}, \tag{2.27}$$

$$= \begin{pmatrix} \mathbf{B}_p^{\psi} & \mathbf{B}_p^{\psi}\mathbf{X}^{\mathrm{T}} & \mathbf{B}_p^{\psi}\mathbf{L}^{\mathrm{T}}\mathbf{G}^{\mathrm{T}} & \mathbf{B}_p^{\psi}\mathbf{L}^{\mathrm{T}}\mathbf{G}^{\mathrm{T}}\mathbf{P}^{\mathrm{T}} \\ \mathbf{X}\mathbf{B}_p^{\psi} & \mathbf{X}\mathbf{B}_p^{\psi}\mathbf{X}^{\mathrm{T}} + \mathbf{B}_p^{\chi^{\mathrm{u}}} & \mathbf{X}\mathbf{B}_p^{\psi}\mathbf{L}^{\mathrm{T}}\mathbf{G}^{\mathrm{T}} & \mathbf{X}\mathbf{B}_p^{\psi}\mathbf{L}^{\mathrm{T}}\mathbf{G}^{\mathrm{T}}\mathbf{P}^{\mathrm{T}} \\ \mathbf{GL}\mathbf{B}_p^{\psi} & \mathbf{GL}\mathbf{B}_p^{\psi}\mathbf{X}^{\mathrm{T}} & \mathbf{GL}\mathbf{B}_p^{\psi}\mathbf{L}^{\mathrm{T}}\mathbf{G}^{\mathrm{T}} + \mathbf{B}_p^{p^{\mathrm{A}}} & \mathbf{GL}\mathbf{B}_p^{\psi}\mathbf{L}^{\mathrm{T}}\mathbf{G}^{\mathrm{T}}\mathbf{P}^{\mathrm{T}} + \mathbf{B}_p^{p^{\mathrm{A}}}\mathbf{P}^{\mathrm{T}} \\ \mathbf{PGL}\mathbf{B}_p^{\psi} & \mathbf{PGL}\mathbf{B}_p^{\psi}\mathbf{X}^{\mathrm{T}} & \mathbf{PGL}\mathbf{B}_p^{\psi}\mathbf{L}^{\mathrm{T}}\mathbf{G}^{\mathrm{T}} + \mathbf{P}\mathbf{B}_p^{p^{\mathrm{A}}} & \mathbf{PGL}\mathbf{B}_p^{\psi}\mathbf{L}^{\mathrm{T}}\mathbf{G}^{\mathrm{T}}\mathbf{P}^{\mathrm{T}} + \mathbf{P}\mathbf{B}_p^{p^{\mathrm{A}}}\mathbf{P}^{\mathrm{T}} \end{pmatrix}. \tag{2.28}$$

3. Option A: Allow for a non-hydrostatic potential temperature (with multi-scale option ii)
This is the first option for use for convective scale data assimilation. In this case, the control variables that are assumed to be uncorrelated are the same control variables as the standard set plus one extra - see below. The balance conditions are switched off in the convective-scale control variable transform. This scheme must be used with multi-scale option ii since using no balance conditions is not a good approach for larger scales. The extra control variable mentioned above is the convective-scale contributions to potential temperature. Even though we include a convective-scale potential temperature (which is not in hydrostatic balance with pressure), all of the pressure increment is still used with the hydrostatic relation.

3.1 The Tp-transform under option A

Input fields	Output fields
δu, δv, δw, $\delta\theta$, δp and δq_{T}	$\delta\psi$, $\delta\chi^{\mathrm{u}}$, δp^{A}, $\delta\theta^{\mathrm{vCS}}$, $\delta\mu$

Tp.1. Calculate the convective-scale component of the virtual potential temperature - see (2.1) and (2.18)

$$\delta\theta^{\mathrm{vCS}} = [1 + (\varepsilon^{-1} - 1) q]\,\delta\theta + \theta\,(\varepsilon^{-1} - 1)\,\delta q_{\mathrm{T}} - \frac{\kappa g}{c_p}\left(\frac{\partial\Pi}{\partial z}\right)^{-2}\frac{\partial}{\partial z}\left(\frac{\Pi^{\mathrm{H}}\delta p}{p^{\mathrm{H}}}\right), \quad (3.1^*)$$

$$\delta\theta^{\mathrm{vCS}} = \Theta^{\theta}\delta\theta + \Theta^{q}\delta q_{\mathrm{T}} - \mathbf{P}\delta p. \quad (3.1a)$$

This calculation is based on the total virtual potential temperature minus the potential temperature that is in hydrostatic balance with pressure.

Tp.2. Calculate the moisture control variable - see (2.3)

$$\delta\mu = a\left(\frac{1}{q_{\mathrm{s}}}\delta q_{\mathrm{T}} - h_1\frac{q\Pi}{q_{\mathrm{s}}}\frac{d\ \ln e_{\mathrm{s}}}{dT}\delta\theta - h_2\frac{q}{pq_{\mathrm{s}}}\left[\kappa\Pi\theta\frac{d\ \ln e_{\mathrm{s}}}{dT} - 1\right]\delta p\right), \quad (3.2^*)$$

$$\delta\mu = \mathbf{M}^{q}\delta q_{\mathrm{T}} + \mathbf{M}^{\theta}\delta\theta + \mathbf{M}^{p}\delta p. \quad (3.2a)$$

Tp.3. Calculate the streamfunction - see (2.4)

$$\delta\psi = \nabla_{\mathrm{h}}^{-2}\left\{\mathbf{k}\cdot\left[\nabla\times\begin{pmatrix}\delta u\\ \delta v\\ 0\end{pmatrix}\right]\right\}, \quad (3.3^*)$$

$$\delta\psi = \mathbf{Y}\delta\mathbf{u}. \quad (3.3a)$$

Tp.4. Calculate the velocity potential - see (2.5)

$$\delta\chi = \nabla_{\mathrm{h}}^{-2}\left\{\nabla\cdot\begin{pmatrix}\delta u\\ \delta v\\ 0\end{pmatrix}\right\}. \quad (3.4)$$

$$\delta\chi = \mathbf{C}\delta\mathbf{u}, \quad (3.4a)$$

Tp.5. Calculate the balanced component of the velocity potential - see (2.6)

$$\delta\chi^{\mathrm{b}} = \mathbf{B}_{\mathrm{v}}^{\chi\psi}\mathbf{B}_{\mathrm{v}}^{\psi\psi^{-1}}\mathbf{S}_{\mathrm{G}}^{\mathrm{LS}}\delta\psi, \quad (3.5)$$

$$= \mathbf{X}\mathbf{S}_{\mathrm{G}}^{\mathrm{LS}}\delta\psi, \tag{3.5a}$$

where $\mathbf{S}_{\mathrm{G}}^{\mathrm{LS}}$ is a filter that allows through only scales where the geostrophic balance approximation is valid. At smaller scales, it is assumed that there is no balanced velocity potential.

Tp.6. Calculate the unbalanced component of the velocity potential - see (2.7)

$$\delta\chi^{\mathrm{u}} = \delta\chi - \delta\chi^{\mathrm{b}}, \tag{3.6* / 3.6a}$$

Tp.7. Calculate the level-by-level geostrophically balanced pressure - see (2.8)

$$\delta p^{\mathrm{G}} = \nabla_{\mathrm{h}}^{-2}\left\{\nabla_{\mathrm{h}} \cdot \left[f\rho\nabla_{\mathrm{h}}\mathbf{S}_{\mathrm{G}}^{\mathrm{LS}}\delta\psi\right]\right\}, \tag{3.7}$$

$$\delta p^{\mathrm{G}} = \mathbf{L}\mathbf{S}_{\mathrm{G}}^{\mathrm{LS}}\delta\psi. \tag{3.7a}$$

Note the presence of the large-scale filter, $\mathbf{S}_{\mathrm{G}}^{\mathrm{LS}}$, which is not present in the standard transforms.

Tp.8. Calculate the vertically regressed geostrophically balanced pressure - see (2.9)

$$\delta p^{\mathrm{F}} = \mathbf{B}_{\mathrm{v}}^{p^{\mathrm{F}}p^{\mathrm{G}}}\mathbf{B}_{\mathrm{v}}^{p^{\mathrm{G}}p^{\mathrm{G}}-1}\delta p^{\mathrm{G}}, \tag{3.8}$$

$$\delta p^{\mathrm{F}} = \mathbf{G}\delta p^{\mathrm{G}}. \tag{3.8a}$$

Tp.9. Calculate the geostrophically unbalanced pressure (ageostrophic pressure) - see (2.11)

$$\delta p^{\mathrm{A}} = \delta p - \delta p^{\mathrm{F}}. \tag{3.9*/3.9a}$$

As in the standard transforms, here the model variables δw and $\delta\rho$ are not used.

3.2 The Up-transform under option A

Input fields	Output fields
$\delta\psi$, $\delta\chi^{\mathrm{u}}$, δp^{A}, $\delta\theta^{\mathrm{vCS}}$, $\delta\mu$	δu, δv, δw, $\delta\theta$, $\delta\rho$, δp and δq_{T}

Up.1. Calculate the balanced velocity potential (as Tp.5)

$$\delta\chi^{\mathrm{b}} = \mathbf{B}_{\mathrm{v}}^{\chi\psi}\mathbf{B}_{\mathrm{v}}^{\psi\psi-1}\mathbf{S}_{\mathrm{G}}^{\mathrm{LS}}\delta\psi, \tag{3.10}$$

$$= \mathbf{X}\mathbf{S}_{\mathrm{G}}^{\mathrm{LS}}\delta\psi. \tag{3.10a}$$

Up.2. Calculate the velocity potential (as Tp.6)

$$\delta\chi = \delta\chi^{\mathrm{u}} + \delta\chi^{\mathrm{b}}. \tag{3.11 / 3.11a}$$

Up.3. Calculate the horizontal wind components (as Tp.3 and Tp.4)

$$\begin{pmatrix}\delta u \\ \delta v\end{pmatrix} = \nabla_{\mathrm{h}}\delta\chi + \mathbf{k} \times \nabla\delta\psi, \tag{3.12*}$$

$$\delta\mathbf{u}_2 = \mathbf{C}^{-1}\delta\chi + \mathbf{Y}^{-1}\delta\psi, \tag{3.12a}$$

where $\delta\mathbf{u}_2$ is a 2-D (horizontal) wind vector.

Up.4. Calculate the level-by-level geostrophically balanced pressure (as Tp.7)

$$\delta p^{\mathrm{G}} = \nabla_{\mathrm{h}}^{-2} \left\{ \nabla_{\mathrm{h}} \cdot \left[f \rho \nabla_{\mathrm{h}} \mathbf{S}_{\mathrm{G}}^{\mathrm{LS}} \delta \psi \right] \right\}, \tag{3.13}$$

$$\delta p^{\mathrm{G}} = \mathbf{L} \mathbf{S}_{\mathrm{G}}^{\mathrm{LS}} \delta \psi. \tag{3.13a}$$

Up.5. Calculate the vertically regressed geostrophically balanced pressure (as Tp.8)

$$\delta p^{\mathrm{F}} = \mathbf{B}_{\mathrm{v}}^{p^{\mathrm{F}} p^{\mathrm{G}}} \mathbf{B}_{\mathrm{v}}^{p^{\mathrm{G}} p^{\mathrm{G}}-1} \delta p^{\mathrm{G}}, \tag{3.14}$$

$$\delta p^{\mathrm{F}} = \mathbf{G} \delta p^{\mathrm{G}}. \tag{3.14a}$$

Up.6. Calculate the pressure (as Tp.9)

$$\delta p = \delta p^{\mathrm{A}} + \delta p^{\mathrm{F}}. \tag{3.15 / 3.15a}$$

Up.7. Calculate the large-scale virtual potential temperature (this operator appears in Tp.1)

$$\delta \theta^{\mathrm{v}^{\mathrm{LS}}} = \frac{\kappa g}{c_p} \left(\frac{\partial \Pi}{\partial z} \right)^{-2} \frac{\partial}{\partial z} \left(\frac{\Pi^{\mathrm{H}} \delta p}{p^{\mathrm{H}}} \right), \tag{3.16}$$

$$\delta \theta^{\mathrm{v}^{\mathrm{LS}}} = \mathbf{P} \delta p, \tag{3.16a}$$

(all pressure is assumed to be hydrostatic).

Up.8. Calculate the total virtual potential temperature (this step is linked to Tp.1)

$$\delta \theta^{\mathrm{v}} = \delta \theta^{\mathrm{v}^{\mathrm{CS}}} + \delta \theta^{\mathrm{v}^{\mathrm{LS}}}. \tag{3.17 / 3.17a}$$

Up.9. Calculate together the potential temperature and the total specific humidity (using the definition of virtual potential temperature, $\delta \theta^{\mathrm{v}} = [1 + (\varepsilon^{-1} - 1) q] \delta \theta + \theta (\varepsilon^{-1} - 1) \delta q_{\mathrm{T}}$ (used in Tp.1, and Tp.2)

$$\delta q_{\mathrm{T}} = \left\{ a h_1 \frac{q \Pi}{q_{\mathrm{s}}} \frac{d \ \ln e_{\mathrm{s}}}{dT} \delta \theta^{\mathrm{v}} + [1 + (\varepsilon^{-1} - 1) q] \delta \mu + [1 + (\varepsilon^{-1} - 1) q] a h_2 \frac{q}{p q_{\mathrm{s}}} \left[\kappa \Pi \theta \frac{d \ \ln e_{\mathrm{s}}}{dT} - 1 \right] \delta p \right\} / \left\{ a h_1 \frac{q \Pi}{q_{\mathrm{s}}} \frac{d \ \ln e_{\mathrm{s}}}{dT} \theta (\varepsilon^{-1} - 1) + [1 + (\varepsilon^{-1} - 1) q] \frac{a}{q_{\mathrm{s}}} \right\}, \tag{3.18*}$$

$$\delta q_{\mathrm{T}} = \mathbf{A}^{-1} \left\{ \Theta^{\theta} \delta \mu - \mathbf{M}^{\theta} \delta \theta^{\mathrm{v}} - \Theta^{\theta} \mathbf{M}^{p} \delta p \right\}, \tag{3.18a}$$

$$\delta \theta = \left\{ \frac{a}{q_{\mathrm{s}}} \delta \theta^{\mathrm{v}} - \theta (\varepsilon^{-1} - 1) \delta \mu - \theta (\varepsilon^{-1} - 1) a h_2 \frac{q}{p q_{\mathrm{s}}} \left[\kappa \Pi \theta \frac{d \ \ln e_{\mathrm{s}}}{dT} - 1 \right] \delta p \right\} / \left\{ a h_1 \frac{q \Pi}{q_{\mathrm{s}}} \frac{d \ \ln e_{\mathrm{s}}}{dT} \theta (\varepsilon^{-1} - 1) + [1 + (\varepsilon^{-1} - 1) q] \frac{a}{q_{\mathrm{s}}} \right\}, \tag{3.19*}$$

$$\delta \theta = \mathbf{A}^{-1} \left\{ \Theta^{q} \delta \mu - \mathbf{M}^{q} \delta \theta^{\mathrm{v}} - \Theta^{q} \mathbf{M}^{p} \delta p \right\}. \tag{3.19a}$$

Up.10. Calculate the density

$$\delta\rho \;=\; \rho\left(\frac{1\,-\,\kappa}{p}\delta p\,-\,\frac{1}{\theta^{\mathrm{v}}}\delta\theta^{\mathrm{v}}\right), \tag{3.20*}$$

$$\delta\rho \;=\; \mathbf{R}^{p}\delta p\,+\,\mathbf{R}^{\theta}\delta\theta^{\mathrm{v}}. \tag{3.20a}$$

Up.11. Calculate the vertical velocity component by solving either the incompressible equation

$$\frac{\partial\,(\delta w\rho_y)}{\partial z}\,+\,\frac{\partial\,(\delta\rho_y w)}{\partial z} \;=\; -\nabla_{\mathrm{h}}\,\cdot\,\left[\rho_y\begin{pmatrix}\delta u\\ \delta v\end{pmatrix}\right]-\,\nabla_{\mathrm{h}}\,\cdot\,\left[\delta\rho_y\begin{pmatrix}u\\ v\end{pmatrix}\right], \tag{3.21*}$$

$$\delta w \;\approx\; \mathbf{W}\delta\mathbf{u}, \tag{3.21a}$$

or by solving Richardson's equation (not shown). For simplicity, the matrix/vector version of (3.21) does not include the $\delta\rho$ contribution.

It has been demonstrated that $\mathbf{T}_p\mathbf{U}_p \;=\; \mathbf{I}$ under the proviso that the total pressure increment is always considered hydrostatic (and thus used with the hydrostatic balance equation).

3.3 The implied background error covariance matrix for option A

We now examine the implied background error covariance matrix for option A. The expressions can become very complicated and so we examine the implied covariances for the variables $\delta\psi$, $\delta\chi$, δp and $\delta\theta^{\mathrm{v}}$ only. Consequently, we need to involve the control variables $\delta\psi$, $\delta\chi^{\mathrm{u}}$, δp^{A} and $\delta\theta^{\mathrm{vCS}}$ only. First, a summary of the Up-transform for this reduced variable set

$$\begin{pmatrix}\delta\psi\\ \delta\chi\\ \delta p\\ \delta\theta^{\mathrm{v}}\end{pmatrix} \;=\; \begin{pmatrix}\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{XS}_{\mathrm{G}}^{\mathrm{LS}} & \mathbf{I} & \mathbf{0} & \mathbf{0}\\ \mathbf{GLS}_{\mathrm{G}}^{\mathrm{LS}} & \mathbf{0} & \mathbf{I} & \mathbf{0}\\ \mathbf{PGLS}_{\mathrm{G}}^{\mathrm{LS}} & \mathbf{0} & \mathbf{P} & \mathbf{I}\end{pmatrix}\begin{pmatrix}\delta\psi\\ \delta\chi^{\mathrm{u}}\\ \delta p^{\mathrm{A}}\\ \delta\theta^{\mathrm{vCS}}\end{pmatrix}. \tag{3.22}$$

The implied covariances are

$$\mathbf{B}^{\mathrm{imp}} \;=\; \mathbf{U}_p\mathbf{B}_p\mathbf{U}_p^{\mathrm{T}}, \tag{3.23}$$

where $\mathbf{B}_p$ is the background error covariance matrix of the variables $\delta\psi$, $\delta\chi^{\mathrm{u}}$, δp^{A} and $\delta\theta^{\mathrm{vCS}}$ (which is block diagonal). This expands to

$$\mathbf{B}^{\mathrm{imp}} \;=\; \begin{pmatrix}\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{XS}_{\mathrm{G}}^{\mathrm{LS}} & \mathbf{I} & \mathbf{0} & \mathbf{0}\\ \mathbf{GLS}_{\mathrm{G}}^{\mathrm{LS}} & \mathbf{0} & \mathbf{I} & \mathbf{0}\\ \mathbf{PGLS}_{\mathrm{G}}^{\mathrm{LS}} & \mathbf{0} & \mathbf{P} & \mathbf{I}\end{pmatrix}\begin{pmatrix}\mathbf{B}_p^{\psi} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{B}_p^{\chi^{\mathrm{u}}} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{B}_p^{p^{\mathrm{A}}} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_p^{\theta^{\mathrm{v}}}\end{pmatrix}\begin{pmatrix}\mathbf{I} & \mathbf{S}_{\mathrm{G}}^{\mathrm{LST}}\mathbf{X}^{\mathrm{T}} & \mathbf{S}_{\mathrm{G}}^{\mathrm{LST}}\mathbf{L}^{\mathrm{T}}\mathbf{G}^{\mathrm{T}} & \mathbf{S}_{\mathrm{G}}^{\mathrm{LST}}\mathbf{L}^{\mathrm{T}}\mathbf{G}^{\mathrm{T}}\mathbf{P}^{\mathrm{T}}\\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{P}^{\mathrm{T}}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}\end{pmatrix},$$

$$= \left(\begin{matrix}\mathbf{B}_p^{\psi} & \mathbf{B}_p^{\psi}\mathbf{S}_{\mathrm{G}}^{\mathrm{LST}}\mathbf{X}^{\mathrm{T}} & \mathbf{B}_p^{\psi}\mathbf{S}_{\mathrm{G}}^{\mathrm{LST}}\mathbf{L}^{\mathrm{T}}\mathbf{G}^{\mathrm{T}}\\ \mathbf{XS}_{\mathrm{G}}^{\mathrm{LS}}\mathbf{B}_p^{\psi} & \mathbf{XS}_{\mathrm{G}}^{\mathrm{LS}}\mathbf{B}_p^{\psi}\mathbf{S}_{\mathrm{G}}^{\mathrm{LST}}\mathbf{X}^{\mathrm{T}}\,+\,\mathbf{B}_p^{\chi^{\mathrm{u}}} & \mathbf{XS}_{\mathrm{G}}^{\mathrm{LS}}\mathbf{B}_p^{\psi}\mathbf{S}_{\mathrm{G}}^{\mathrm{LST}}\mathbf{L}^{\mathrm{T}}\mathbf{G}^{\mathrm{T}}\\ \mathbf{GLS}_{\mathrm{G}}^{\mathrm{LS}}\mathbf{B}_p^{\psi} & \mathbf{GLS}_{\mathrm{G}}^{\mathrm{LS}}\mathbf{B}_p^{\psi}\mathbf{S}_{\mathrm{G}}^{\mathrm{LST}}\mathbf{X}^{\mathrm{T}} & \mathbf{GLS}_{\mathrm{G}}^{\mathrm{LS}}\mathbf{B}_p^{\psi}\mathbf{S}_{\mathrm{G}}^{\mathrm{LST}}\mathbf{L}^{\mathrm{T}}\mathbf{G}^{\mathrm{T}}\,+\,\mathbf{B}_p^{p^{\mathrm{A}}}\\ \mathbf{PGLS}_{\mathrm{G}}^{\mathrm{LS}}\mathbf{B}_p^{\psi} & \mathbf{PGLS}_{\mathrm{G}}^{\mathrm{LS}}\mathbf{B}_p^{\psi}\mathbf{S}_{\mathrm{G}}^{\mathrm{LST}}\mathbf{X}^{\mathrm{T}} & \mathbf{PGLS}_{\mathrm{G}}^{\mathrm{LS}}\mathbf{B}_p^{\psi}\mathbf{S}_{\mathrm{G}}^{\mathrm{LST}}\mathbf{L}^{\mathrm{T}}\mathbf{G}^{\mathrm{T}}\,+\,\mathbf{PB}_p^{p^{\mathrm{A}}}\end{matrix}\right. \tag{3.24}$$

$$\left.\begin{array}{r}\mathbf{B}_p^{\psi}\mathbf{S}_{\mathrm{G}}^{\mathrm{LS}\,\mathrm{T}}\mathbf{L}^{\mathrm{T}}\mathbf{G}^{\mathrm{T}}\mathbf{P}^{\mathrm{T}} \\ \mathbf{X}\mathbf{S}_{\mathrm{G}}^{\mathrm{LS}}\mathbf{B}_p^{\psi}\mathbf{S}_{\mathrm{G}}^{\mathrm{LS}\,\mathrm{T}}\mathbf{L}^{\mathrm{T}}\mathbf{G}^{\mathrm{T}}\mathbf{P}^{\mathrm{T}} \\ \mathbf{G}\mathbf{L}\mathbf{S}_{\mathrm{G}}^{\mathrm{LS}}\mathbf{B}_p^{\psi}\mathbf{S}_{\mathrm{G}}^{\mathrm{LS}\,\mathrm{T}}\mathbf{L}^{\mathrm{T}}\mathbf{G}^{\mathrm{T}}\mathbf{P}^{\mathrm{T}} + \mathbf{B}_p^{p\mathrm{A}}\mathbf{P}^{\mathrm{T}} \\ \mathbf{P}\mathbf{G}\mathbf{L}\mathbf{S}_{\mathrm{G}}^{\mathrm{LS}}\mathbf{B}_p^{\psi}\mathbf{S}_{\mathrm{G}}^{\mathrm{LS}\,\mathrm{T}}\mathbf{L}^{\mathrm{T}}\mathbf{G}^{\mathrm{T}}\mathbf{P}^{\mathrm{T}} + \mathbf{P}\mathbf{B}_p^{p\mathrm{A}}\mathbf{P}^{\mathrm{T}} + \mathbf{B}_p^{\theta\mathrm{v}}\end{array}\right). \quad (3.25)$$

This result can now be compared to (2.28) for the standard scheme. The only differences are the inclusion of the scale filter $\mathbf{S}_{\mathrm{G}}^{\mathrm{LS}}$ and the modification to the virtual potential temperature covariances with $\mathbf{B}_p^{\theta\mathrm{v}}$ (some components of the $\mathbf{B}_p$-matrix will be different between the schemes - even for variables that have the same name in each scheme).

4. Option B: Allow for a non-hydrostatic potential temperature and non-hydrostatic pressure

A covariance model based on balances that are not valid at convective-scales (such as geostrophic and/or hydrostatic balance) is likely to lead to analysis increments that are inappropriate. In a conventional data assimilation system which invokes geostrophic and hydrostatic balances, a single observation will give rise to an innovation that will adjust the fields non-locally according to these balances. These include non-local increments to the mass and horizontal wind fields and adjustments in the vertical to maintain hydrostatic balance. In the event that the innovation has arisen due to forecast errors of a convective system that is ageostrophic and non-hydrostatic, these analysis increments would be inappropriate and may, e.g., remove convective-scale features that are important. The balance constraints may be lifted by use of option A above, but which still makes the assumption that all pressure is hydrostatically balanced. Although an improvement on the standard scheme, this scheme may still be significantly sub-optimal when used for the convective-scale problem.

Anelastic balance (Pielke, 2002) is an alternative balance that we would like to investigate for the convective-scale data assimilation problem. Subsections 4.1 to 4.7 are concerned with developing equations associated with anelastic balance.

4.1 Field decomposition

In order to study anelastic balance in a multi-scale system, we need to first introduce the following notation for the decomposition of fields

$$\phi = \bar{\phi} + \phi', \tag{4.1}$$

$$\bar{\phi} = \phi^{\mathrm{LS}} + \phi^{\mathrm{CS}}, \tag{4.2}$$

where ϕ is a generic atmospheric variable. In (4.1), ϕ is the value of the field at a particular position and time which is decomposed into grid-box mean, $\bar{\phi}$, and sub-grid-scale, ϕ', parts. In (4.2) the grid-box-mean is itself decomposed into large-scale, ϕ^{LS}, and convective-scale, ϕ^{CS}, parts. The large-scale part is assumed to be in hydrostatic balance. Pielke (2002, Sec. 4.1) makes a similar decomposition, but here we use a slightly different notation for clarity. In incremental data assimilation, we have a reference state (e.g. the background) and an increment. Incremental quantities are decomposed as (4.2) (there are no sub-grid-scale data assimilation increments), but are preceded by a δ (as before)

$$\delta\bar{\phi} = \delta\phi^{\mathrm{LS}} + \delta\phi^{\mathrm{CS}}. \tag{4.3}$$

4.2 Momentum equations for the grid-box-mean variables

[Note that readers wishing to skip the derivation of the anelastic equations may go straight to the result (4.26).] Pielke (2002, Eqs. (4.21)) gives the equations of motion for the grid-box-mean winds (by Reynolds averaging), which provide the starting point for this discussion

$$\begin{aligned}\frac{\partial \bar{u}}{\partial t} &= -\bar{u}\frac{\partial \bar{u}}{\partial x} - \bar{v}\frac{\partial \bar{u}}{\partial y} - \bar{w}\frac{\partial \bar{u}}{\partial z} \\ &\quad -\alpha^{\mathrm{LS}}\frac{\partial}{\partial x}\rho^{\mathrm{LS}}\overline{u'u'} - \alpha^{\mathrm{LS}}\frac{\partial}{\partial y}\rho^{\mathrm{LS}}\overline{v'u'} - \alpha^{\mathrm{LS}}\frac{\partial}{\partial z}\rho^{\mathrm{LS}}\overline{w'u'} - \alpha^{\mathrm{LS}}\frac{\partial\left(p^{\mathrm{CS}} + p^{\mathrm{LS}}\right)}{\partial x} + f\bar{v}, \end{aligned} \tag{4.4}$$

$$\frac{\partial \bar{v}}{\partial t} = -\bar{u}\frac{\partial \bar{v}}{\partial x} - \bar{v}\frac{\partial \bar{v}}{\partial y} - \bar{w}\frac{\partial \bar{v}}{\partial z}$$

$$-\alpha^{LS}\frac{\partial}{\partial x}\rho^{LS}\overline{u'v'} - \alpha^{LS}\frac{\partial}{\partial y}\rho^{LS}\overline{v'v'} - \alpha^{LS}\frac{\partial}{\partial z}\rho^{LS}\overline{w'v'} - \alpha^{LS}\frac{\partial\left(p^{CS} + p^{LS}\right)}{\partial y} - f\bar{u}, \quad (4.5)$$

$$\frac{\partial\bar{w}}{\partial t} = -\bar{u}\frac{\partial\bar{w}}{\partial x} - \bar{v}\frac{\partial\bar{w}}{\partial y} - \bar{w}\frac{\partial\bar{w}}{\partial z}$$
$$-\alpha^{LS}\frac{\partial}{\partial x}\rho^{LS}\overline{u'w'} - \alpha^{LS}\frac{\partial}{\partial y}\rho^{LS}\overline{v'w'} - \alpha^{LS}\frac{\partial}{\partial z}\rho^{LS}\overline{w'w'} - \alpha^{LS}\frac{\partial p^{CS}}{\partial z} + g\frac{\alpha^{CS}}{\alpha^{LS}}, \quad (4.6)$$

where

$$\alpha = \frac{1}{\rho}, \quad (4.7)$$

is the specific volume. Many terms are kept in these equations at this stage, for instance the sub-grid-scale momentum fluxes. These equations, and the decomposition (4.1)-(4.3), will result in some relatively involved, but straightforward, algebra. The large-scale fields are taken to be in hydrostatic balance

$$\alpha^{LS}\frac{\partial p^{LS}}{\partial z} = -g. \quad (4.8)$$

The Boussinesq approximation has been used to give (4.6) which can be understood as follows. Consider $\bar{\alpha}\partial\bar{p}/\partial z + g$ when decomposed into large- and convective-scales

$$\begin{aligned}
\bar{\alpha}\frac{\partial\bar{p}}{\partial z} + g &= \left(\alpha^{LS} + \alpha^{CS}\right)\frac{\partial\left(p^{LS} + p^{CS}\right)}{\partial z} + g, \\
&= \alpha^{LS}\frac{\partial\left(p^{LS} + p^{CS}\right)}{\partial z} + \alpha^{CS}\frac{\partial\left(p^{LS} + p^{CS}\right)}{\partial z} + g, \\
&= \alpha^{LS}\frac{\partial p^{CS}}{\partial z} + \alpha^{CS}\frac{\partial\left(p^{LS} + p^{CS}\right)}{\partial z}, \\
&= \alpha^{LS}\frac{\partial p^{CS}}{\partial z} - g\frac{\alpha^{CS}}{\alpha^{LS}} + \alpha^{CS}\frac{\partial p^{CS}}{\partial z}, \\
&= \left(\alpha^{LS} + \alpha^{CS}\right)\frac{\partial p^{CS}}{\partial z} - g\frac{\alpha^{CS}}{\alpha^{LS}}, \\
&\approx \alpha^{LS}\frac{\partial p^{CS}}{\partial z} - g\frac{\alpha^{CS}}{\alpha^{LS}}. \quad (4.9)
\end{aligned}$$

It is the approximation made to give the last line (that $\alpha^{CS} \ll \alpha^{LS}$ except when multiplying g) that is the Boussinesq approximation.

4.3 Three-dimensional divergence of the momentum equations

As done in Pielke (2002, Secs. 4.2 and 4.3), multiply (4.4)-(4.6) by ρ^{LS} (assume that $\partial\rho^{LS}/\partial t$ is negligible) and calculate the 3-D divergence

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{\partial\rho^{LS}\bar{u}}{\partial x} + \frac{\partial\rho^{LS}\bar{v}}{\partial y} + \frac{\partial\rho^{LS}\bar{w}}{\partial z}\right) = &-\frac{\partial}{\partial x}\left(\rho^{LS}\bar{u}\frac{\partial\bar{u}}{\partial x}\right) - \frac{\partial}{\partial y}\left(\rho^{LS}\bar{u}\frac{\partial\bar{v}}{\partial x}\right) - \frac{\partial}{\partial z}\left(\rho^{LS}\bar{u}\frac{\partial\bar{w}}{\partial x}\right) \\
&-\frac{\partial}{\partial x}\left(\rho^{LS}\bar{v}\frac{\partial\bar{u}}{\partial y}\right) - \frac{\partial}{\partial y}\left(\rho^{LS}\bar{v}\frac{\partial\bar{v}}{\partial y}\right) - \frac{\partial}{\partial z}\left(\rho^{LS}\bar{v}\frac{\partial\bar{w}}{\partial y}\right) \\
&-\frac{\partial}{\partial x}\left(\rho^{LS}\bar{w}\frac{\partial\bar{u}}{\partial z}\right) - \frac{\partial}{\partial y}\left(\rho^{LS}\bar{w}\frac{\partial\bar{v}}{\partial z}\right) - \frac{\partial}{\partial z}\left(\rho^{LS}\bar{w}\frac{\partial\bar{w}}{\partial z}\right)
\end{aligned}$$

$$
\begin{aligned}
&-\frac{\partial}{\partial x}\frac{\partial}{\partial x}\rho^{\mathrm{LS}}\overline{u'u'} - \frac{\partial}{\partial y}\frac{\partial}{\partial x}\rho^{\mathrm{LS}}\overline{u'v'} - \frac{\partial}{\partial z}\frac{\partial}{\partial x}\rho^{\mathrm{LS}}\overline{u'w'} \\
&-\frac{\partial}{\partial x}\frac{\partial}{\partial y}\rho^{\mathrm{LS}}\overline{v'u'} - \frac{\partial}{\partial y}\frac{\partial}{\partial y}\rho^{\mathrm{LS}}\overline{v'v'} - \frac{\partial}{\partial z}\frac{\partial}{\partial y}\rho^{\mathrm{LS}}\overline{v'w'} \\
&-\frac{\partial}{\partial x}\frac{\partial}{\partial z}\rho^{\mathrm{LS}}\overline{w'u'} - \frac{\partial}{\partial y}\frac{\partial}{\partial z}\rho^{\mathrm{LS}}\overline{w'v'} - \frac{\partial}{\partial z}\frac{\partial}{\partial z}\rho^{\mathrm{LS}}\overline{w'w'} \\
&-\frac{\partial^2 (p^{\mathrm{CS}} + p^{\mathrm{LS}})}{\partial x^2} - \frac{\partial^2 (p^{\mathrm{CS}} + p^{\mathrm{LS}})}{\partial y^2} - \frac{\partial^2 p^{\mathrm{CS}}}{\partial z^2} \\
&+\frac{\partial \rho^{\mathrm{LS}} f \bar{v}}{\partial x} - \frac{\partial \rho^{\mathrm{LS}} f \bar{u}}{\partial y} + g\frac{\partial}{\partial z}\left(\rho^{\mathrm{LS}}\frac{\alpha^{\mathrm{CS}}}{\alpha^{\mathrm{LS}}}\right),
\end{aligned}
\tag{4.10}
$$

$$
\begin{aligned}
\frac{\partial}{\partial t}\nabla\cdot(\rho^{\mathrm{LS}}\bar{\mathbf{u}}) = &-\frac{\partial}{\partial x}\rho^{\mathrm{LS}}\bar{\mathbf{u}}\cdot\nabla\bar{u} - \frac{\partial}{\partial y}\rho^{\mathrm{LS}}\bar{\mathbf{u}}\cdot\nabla\bar{v} - \frac{\partial}{\partial z}\rho^{\mathrm{LS}}\bar{\mathbf{u}}\cdot\nabla\bar{w} \\
&-\frac{\partial}{\partial x}\nabla\cdot\rho^{\mathrm{LS}}\overline{\mathbf{u}'u'} - \frac{\partial}{\partial y}\nabla\cdot\rho^{\mathrm{LS}}\overline{\mathbf{u}'v'} - \frac{\partial}{\partial z}\nabla\cdot\rho^{\mathrm{LS}}\overline{\mathbf{u}'w'} \\
&-\nabla_{\mathrm{h}}^2 p^{\mathrm{LS}} - \nabla^2 p^{\mathrm{CS}} + \mathbf{k}\cdot\nabla\times\rho^{\mathrm{LS}} f\bar{\mathbf{u}} + g\frac{\partial}{\partial z}\left(\rho^{\mathrm{LS}}\frac{\alpha^{\mathrm{CS}}}{\alpha^{\mathrm{LS}}}\right).
\end{aligned}
\tag{4.11}
$$

Equation (4.11) is the same as (4.10), but in a compact notation where ∇_{h} comprises the horizontal components of the gradient vector and ∇ comprises all three components of the gradient vector.

4.4 Anelastic balance and a diagnostic equation for convective-scale pressure

The anelastic approximation states that

$$
\frac{\partial}{\partial t}\nabla\cdot(\rho^{\mathrm{LS}}\bar{\mathbf{u}}) = 0. \tag{4.12}
$$

This gives a diagnostic equation for convective-scale pressure p^{CS}

$$
\begin{aligned}
\nabla^2 p^{\mathrm{CS}} - g\frac{\partial}{\partial z}\left(\rho^{\mathrm{LS}}\frac{\alpha^{\mathrm{CS}}}{\alpha^{\mathrm{LS}}}\right) = &-\frac{\partial}{\partial x}\rho^{\mathrm{LS}}\bar{\mathbf{u}}\cdot\nabla\bar{u} - \frac{\partial}{\partial y}\rho^{\mathrm{LS}}\bar{\mathbf{u}}\cdot\nabla\bar{v} - \frac{\partial}{\partial z}\rho^{\mathrm{LS}}\bar{\mathbf{u}}\cdot\nabla\bar{w} \\
&-\frac{\partial}{\partial x}\nabla\cdot\rho^{\mathrm{LS}}\overline{\mathbf{u}'u'} - \frac{\partial}{\partial y}\nabla\cdot\rho^{\mathrm{LS}}\overline{\mathbf{u}'v'} - \frac{\partial}{\partial z}\nabla\cdot\rho^{\mathrm{LS}}\overline{\mathbf{u}'w'} \\
&-\nabla_{\mathrm{h}}^2 p^{\mathrm{LS}} + \mathbf{k}\cdot\nabla_3\times\rho^{\mathrm{LS}} f\bar{\mathbf{u}}.
\end{aligned}
\tag{4.13}
$$

The convective-scale term α^{CS} has been put on the left hand side because it has a dependency upon p^{CS} via the ideal gas law. The ideal gas law may be developed as follows

$$
p\alpha = RT^{\mathrm{v}},
$$

$$
\ln p + \ln\alpha = \ln R + \ln T^{\mathrm{v}},
$$

which may be linearized as follows

$$
\frac{dp}{p} + \frac{d\alpha}{\alpha} = \frac{dT^{\mathrm{v}}}{T^{\mathrm{v}}}. \tag{4.14}
$$

When the linearization state comprises the large-scale fields and the convective-scale components are small, (4.14) leads to the following

$$\frac{p^{\mathrm{CS}}}{p^{\mathrm{LS}}} + \frac{\alpha^{\mathrm{CS}}}{\alpha^{\mathrm{LS}}} = \frac{T^{\mathrm{vCS}}}{T^{\mathrm{vLS}}}. \tag{4.15}$$

This can be written in terms of potential temperature, $T^{\mathrm{v}} = (p/p_{1000})^{\kappa}\theta^{\mathrm{v}}$, which may be linearized as follows

$$\ln T^{\mathrm{v}} = \ln\theta^{\mathrm{v}} + \kappa\left(\ln p - \ln 1000\right),$$
$$\frac{T^{\mathrm{vCS}}}{T^{\mathrm{vLS}}} = \frac{\theta^{\mathrm{vCS}}}{\theta^{\mathrm{vLS}}} + \kappa\frac{p^{\mathrm{CS}}}{p^{\mathrm{LS}}}. \tag{4.16}$$

Eliminating T^{vCS} between (4.15) and (4.16) gives

$$\frac{p^{\mathrm{CS}}}{p^{\mathrm{LS}}} + \frac{\alpha^{\mathrm{CS}}}{\alpha^{\mathrm{LS}}} = \frac{\theta^{\mathrm{vCS}}}{\theta^{\mathrm{vLS}}} + \kappa\frac{p^{\mathrm{CS}}}{p^{\mathrm{LS}}},$$
$$\frac{\alpha^{\mathrm{CS}}}{\alpha^{\mathrm{LS}}} = \frac{\theta^{\mathrm{vCS}}}{\theta^{\mathrm{vLS}}} + (\kappa - 1)\frac{p^{\mathrm{CS}}}{p^{\mathrm{LS}}},$$
$$\frac{\alpha^{\mathrm{CS}}}{\alpha^{\mathrm{LS}}} = \frac{\theta^{\mathrm{vCS}}}{\theta^{\mathrm{vLS}}} - \frac{c_v}{c_p}\frac{p^{\mathrm{CS}}}{p^{\mathrm{LS}}}, \tag{4.17}$$

where $\kappa - 1 = -c_v/c_p$. Substituting (4.17) into (4.13) allows the convective-scale pressure contribution to be separated from the other variables

$$\nabla^2 p^{\mathrm{CS}} + g\frac{c_v}{c_p}\frac{\partial}{\partial z}\left(\rho^{\mathrm{LS}}\frac{p^{\mathrm{CS}}}{p^{\mathrm{LS}}}\right) = -\frac{\partial}{\partial x}\rho^{\mathrm{LS}}\bar{\mathbf{u}}\cdot\nabla\bar{u} - \frac{\partial}{\partial y}\rho^{\mathrm{LS}}\bar{\mathbf{u}}\cdot\nabla\bar{v} - \frac{\partial}{\partial z}\rho^{\mathrm{LS}}\bar{\mathbf{u}}\cdot\nabla\bar{w}$$
$$-\frac{\partial}{\partial x}\nabla\cdot\rho^{\mathrm{LS}}\overline{\mathbf{u}'u'} - \frac{\partial}{\partial y}\nabla\cdot\rho^{\mathrm{LS}}\overline{\mathbf{u}'v'} - \frac{\partial}{\partial z}\nabla\cdot\rho^{\mathrm{LS}}\overline{\mathbf{u}'w'}$$
$$-\nabla_{\mathrm{h}}^2 p^{\mathrm{LS}} + \mathbf{k}\cdot\nabla\times\rho^{\mathrm{LS}} f\bar{\mathbf{u}} + g\frac{\partial}{\partial z}\left(\rho^{\mathrm{LS}}\frac{\theta^{\mathrm{vCS}}}{\theta^{\mathrm{vLS}}}\right). \tag{4.18}$$

This may be written in the following compact form (the sub-grid-scale terms are ignored from now on - there may scope in later work to include them in a parametrised form - and the overbar notation is dropped)

$$\nabla^2 p^{\mathrm{CS}} + g\frac{c_v}{c_p}\frac{\partial}{\partial z}\left(\frac{1}{\alpha^{\mathrm{LS}}}\frac{p^{\mathrm{CS}}}{p^{\mathrm{LS}}}\right) = -\nabla\cdot\left(\frac{1}{\alpha^{\mathrm{LS}}}(\mathbf{u}\cdot\nabla_3)\,\mathbf{u}\right)$$
$$-\nabla_{\mathrm{h}}^2 p^{\mathrm{LS}} + \mathbf{k}\cdot\nabla\times\left(\frac{f}{\alpha^{\mathrm{LS}}}\mathbf{u}\right) + g\frac{\partial}{\partial z}\left(\frac{1}{\alpha^{\mathrm{LS}}}\frac{{}^{\mathrm{v}}\theta^{\mathrm{CS}}}{\theta^{\mathrm{vLS}}}\right). \tag{4.19}$$

This is a diagnostic equation for p^{CS}.

4.5 Comment on the diagnostic pressure equation

A state that happens to satisfy hydrostatic balance

$$\alpha\frac{\partial p}{\partial z} = -g, \tag{4.20}$$

has the following linearization

$$\alpha\frac{\partial dp}{\partial z} + \frac{\partial p}{\partial z}d\alpha = 0. \tag{4.21}$$

When the linearization state comprises the large-scale fields and the convective-scale components are small, (4.21) leads to the following

$$\alpha^{\mathrm{LS}} \frac{\partial p^{\mathrm{CS}}}{\partial z} + \frac{\partial p^{\mathrm{LS}}}{\partial z} \alpha^{\mathrm{CS}} = 0,$$

$$\alpha^{\mathrm{LS}} \frac{\partial p^{\mathrm{CS}}}{\partial z} = g \frac{\alpha^{\mathrm{CS}}}{\alpha^{\mathrm{LS}}},$$

$$= g \frac{\theta^{\mathrm{vCS}}}{\theta^{\mathrm{vLS}}} - g \frac{c_v}{c_p} \frac{p^{\mathrm{CS}}}{p^{\mathrm{LS}}}, \tag{4.22}$$

where (4.17) has been used in the last line. Multiplying by ρ^{LS} and differentiating with respect to z gives

$$\frac{\partial^2 p^{\mathrm{CS}}}{\partial z^2} = g \frac{\partial}{\partial z} \left(\rho^{\mathrm{LS}} \frac{\theta^{\mathrm{vCS}}}{\theta^{\mathrm{vLS}}} \right) - g \frac{c_v}{c_p} \frac{\partial}{\partial z} \left(\rho^{\mathrm{LS}} \frac{p^{\mathrm{CS}}}{p^{\mathrm{LS}}} \right). \tag{4.23}$$

Substituting this into (4.19) and ignoring advection terms leaves

$$\nabla^2 p^{\mathrm{CS}} - \frac{\partial^2 p^{\mathrm{CS}}}{\partial z^2} = -\nabla_{\mathrm{h}}^2 p^{\mathrm{LS}} + \mathbf{k} \cdot \nabla \times \rho^{\mathrm{LS}} f \mathbf{u},$$

$$\nabla_{\mathrm{h}}^2 p^{\mathrm{CS}} = -\nabla_{\mathrm{h}}^2 p^{\mathrm{LS}} + \mathbf{k} \cdot \nabla \times \rho^{\mathrm{LS}} f \mathbf{u},$$

$$\nabla_{\mathrm{h}}^2 p = \mathbf{k} \cdot \nabla \times \rho^{\mathrm{LS}} f \mathbf{u}, \tag{4.24}$$

which is geostrophic balance. Therefore when anelastic balance holds (and when the advection terms can be ignored), there appears to be a mutual correspondence between the satisfaction of geostrophic and hydrostatic balances. It is not yet clear how useful this observation might be in the design of a control variable transform.

4.6 Strategy for the design of a diagnostic relationship for use in incremental data assimilation

In the incremental data assimilation method, an increment is added to a reference state at the level of grid-box-mean quantities. The increments are specified according to the notation in Sec. 4.1 (from now on though sub-grid-scale quantities will be neglected and the overbars will be dropped since all quantities will exist at the level of grid-box-mean). Large-scale quantities (reference and incremental states) will be assumed to be in exact hydrostatic balance, but not necessarily in exact geostrophic balance.

Data assimilation increments are introduced by linearization of (4.19). There are a number of strategies and each leads to a possible way that the convective-scale assimilation problem may be solved. One is considered below and another is considered in Sec. 5.

4.7 Linearization #1 of the anelastic diagnostic equation

Equation (4.19) is linearized as follows

$$\nabla^2 \delta p^{\mathrm{CS}} - g \frac{c_v}{c_p} \frac{\partial}{\partial z} \left(\frac{\delta \alpha^{\mathrm{LS}}}{\alpha^{\mathrm{LS}2}} \frac{p^{\mathrm{CS}}}{p^{\mathrm{LS}}} \right) + g \frac{c_v}{c_p} \frac{\partial}{\partial z} \left(\frac{1}{\alpha^{\mathrm{LS}}} \frac{\delta p^{\mathrm{CS}}}{p^{\mathrm{LS}}} \right) - g \frac{c_v}{c_p} \frac{\partial}{\partial z} \left(\frac{1}{\alpha^{\mathrm{LS}}} \frac{p^{\mathrm{CS}}}{p^{\mathrm{LS}2}} \delta p^{\mathrm{LS}} \right) =$$

$$\nabla \cdot \left(\frac{\delta \alpha^{\mathrm{LS}}}{\alpha^{\mathrm{LS}2}} (\mathbf{u} \cdot \nabla) \mathbf{u} \right) - \nabla \cdot \left(\frac{1}{\alpha^{\mathrm{LS}}} (\delta \mathbf{u} \cdot \nabla) \mathbf{u} \right) - \nabla \cdot \left(\frac{1}{\alpha^{\mathrm{LS}}} (\mathbf{u} \cdot \nabla) \delta \mathbf{u} \right)$$

$$-\nabla_{\mathrm{h}}^2 \delta p^{\mathrm{LS}} - \mathbf{k} \cdot \nabla \times \left(f \frac{\delta \alpha^{\mathrm{LS}}}{\alpha^{\mathrm{LS}2}} \mathbf{u} \right) + \mathbf{k} \cdot \nabla \times \left(\frac{f}{\alpha^{\mathrm{LS}}} \delta \mathbf{u} \right)$$

$$-g\frac{\partial}{\partial z}\left(\frac{\delta\alpha^{\mathrm{LS}}}{\alpha^{\mathrm{LS}2}}\frac{\theta^{\mathrm{vCS}}}{\theta^{\mathrm{vLS}}}\right) + g\frac{\partial}{\partial z}\left(\frac{1}{\alpha^{\mathrm{LS}}}\frac{\delta\theta^{\mathrm{vCS}}}{\theta^{\mathrm{vLS}}}\right) - g\frac{\partial}{\partial z}\left(\frac{1}{\alpha^{\mathrm{LS}}}\frac{\theta^{\mathrm{vCS}}}{\theta^{\mathrm{vLS}2}}\delta\theta^{\mathrm{vLS}}\right). \qquad (4.25)$$

In the above there are large-scale increments ($\delta\alpha^{\mathrm{LS}}$, δp^{LS} and $\delta\theta^{\mathrm{vLS}}$), convective-scale increments (δp^{CS} and $\delta\theta^{\mathrm{CS}}$) and general increments $\delta\mathbf{u}$. If the large-scale increments are known separately (see e.g. multi-scale option ii in Sec. 1), then (4.25) is a diagnostic equation relating δp^{CS} and $\delta\theta^{\mathrm{vCS}}$. This means that a convective-scale control variable transform based on (4.25) would not require either a pressure-related or a temperature-related control variable. For example, if $\delta\theta^{\mathrm{vCS}}$ is an extra control variable associated with convective-scales, then δp^{CS} could be diagnosed from (4.25). For the convective-scales this would effectively replace the hydrostatic balance step in the current scheme, e.g. (2.2), with the solution of (4.25) for δp^{CS} (although hydrostatic balance would still be used at large-scales).

In (4.25), the large-scale increments are $\delta\alpha^{\mathrm{LS}}$, δp^{LS} and $\delta\theta^{\mathrm{vLS}}$, but recall that the usual (i.e. large-scale) control variables are $\delta\psi$, $\delta\chi^{\mathrm{u}}$, δp^{A} and $\delta\mu$. Let us assume that we can, as part of a grand control variable transform, diagnose the former large-scale increments from the standard control variables whilst maintaining a large/convective-scale separation. Then $\delta\alpha^{\mathrm{LS}}$, δp^{LS}, $\delta\theta^{\mathrm{vLS}}$ and $\delta\mathbf{u}$, would lead to the following rearrangement of (4.25)

$$\begin{aligned}
&\left\{\nabla^2 \bullet \ + g\frac{c_v}{c_p}\frac{\partial}{\partial z}\left(\frac{1}{\alpha^{\mathrm{LS}}}\frac{\bullet}{p^{\mathrm{LS}}}\right)\right\}\delta p^{\mathrm{CS}} = \\
&\quad\left\{g\frac{c_v}{c_p}\frac{\partial}{\partial z}\left(\frac{\bullet}{\alpha^{\mathrm{LS}2}}\frac{p^{\mathrm{CS}}}{p^{\mathrm{LS}}}\right) + \nabla\cdot\left(\frac{\bullet}{\alpha^{\mathrm{LS}2}}(\mathbf{u}\cdot\nabla)\mathbf{u}\right) - \mathbf{k}\cdot\nabla\times\left(f\frac{\bullet}{\alpha^{\mathrm{LS}2}}\mathbf{u}\right) - g\frac{\partial}{\partial z}\left(\frac{\bullet}{\alpha^{\mathrm{LS}2}}\frac{\theta^{\mathrm{vCS}}}{\theta^{\mathrm{vLS}}}\right)\right\}\delta\alpha^{\mathrm{LS}} \\
&\quad+\left\{g\frac{c_v}{c_p}\frac{\partial}{\partial z}\left(\frac{1}{\alpha^{\mathrm{LS}}}\frac{p^{\mathrm{CS}}}{p^{\mathrm{LS}2}}\bullet\right) - \nabla_{\mathrm{h}}^2\bullet\right\}\delta p^{\mathrm{LS}} \\
&\quad-g\frac{\partial}{\partial z}\left(\frac{1}{\alpha^{\mathrm{LS}}}\frac{\theta^{\mathrm{vCS}}}{\theta^{\mathrm{vLS}2}}\bullet\right)\delta\theta^{\mathrm{vLS}} \\
&\quad+g\frac{\partial}{\partial z}\left(\frac{1}{\alpha^{\mathrm{LS}}}\frac{\bullet}{\theta^{\mathrm{vLS}}}\right)\delta\theta^{\mathrm{vCS}} \\
&\quad+\left\{-\nabla\cdot\left(\frac{1}{\alpha^{\mathrm{LS}}}(\bullet\cdot\nabla)\mathbf{u}\right) - \nabla\cdot\left(\frac{1}{\alpha^{\mathrm{LS}}}(\mathbf{u}\cdot\nabla)\bullet\right) + \mathbf{k}\cdot\nabla\times\left(\frac{f}{\alpha^{\mathrm{LS}}}\bullet\right)\right\}\delta\mathbf{u},
\end{aligned} \qquad (4.26)$$

$$\begin{aligned}
\Pi\delta p^{\mathrm{CS}} &= \mathbf{P}^{\alpha}\delta\alpha^{\mathrm{LS}} + \mathbf{P}^{p}\delta p^{\mathrm{LS}} + \mathbf{P}^{\theta\mathrm{LS}}\delta\theta^{\mathrm{vLS}} + \mathbf{P}^{\theta\mathrm{CS}}\delta\theta^{\mathrm{vCS}} + \mathbf{P}^{u}\delta\mathbf{u}, && (4.26a)\\
&= \mathbf{P}^{\alpha}\delta\alpha^{\mathrm{LS}} + \mathbf{P}^{p}\delta p^{\mathrm{LS}} + \mathbf{P}^{\theta\mathrm{LS}}\delta\theta^{\mathrm{vLS}} + \mathbf{P}^{\theta\mathrm{CS}}\delta\theta^{\mathrm{vCS}} + \mathbf{P}_{\mathrm{h}}^{u}\delta\mathbf{u}_2 + \mathbf{P}_{\mathrm{v}}^{u}\delta w, \\
&= \mathbf{P}^{\alpha}\delta\alpha^{\mathrm{LS}} + \mathbf{P}^{p}\delta p^{\mathrm{LS}} + \mathbf{P}^{\theta\mathrm{LS}}\delta\theta^{\mathrm{vLS}} + \mathbf{P}^{\theta\mathrm{CS}}\delta\theta^{\mathrm{vCS}} + (\mathbf{P}_{\mathrm{h}}^{u} + \mathbf{P}_{\mathrm{v}}^{u}\mathbf{W})\delta\mathbf{u}_2, \\
&= \mathbf{P}^{\alpha}\delta\alpha^{\mathrm{LS}} + \mathbf{P}^{p}\delta p^{\mathrm{LS}} + \mathbf{P}^{\theta\mathrm{LS}}\delta\theta^{\mathrm{vLS}} + \mathbf{P}^{\theta\mathrm{CS}}\delta\theta^{\mathrm{vCS}} + \mathbf{P}_{2}^{u}\delta\mathbf{u}_2, && (4.26b)
\end{aligned}$$

Equation (4.26a) has been developed into (4.26b) by first splitting the $\mathbf{P}^{u}$ operator into horizontal ($\mathbf{P}_{\mathrm{h}}^{u}$) and vertical ($\mathbf{P}_{\mathrm{v}}^{u}$) parts, then writing the vertical wind in terms of the horizontal wind using (3.21a). The operator $\mathbf{P}_{2}^{u}$ is defined as $\mathbf{P}_{2}^{u} = \mathbf{P}_{\mathrm{h}}^{u} + \mathbf{P}_{\mathrm{v}}^{u}\mathbf{W}$. Forms (4.26a) and (4.26b) are both used in the rest of this document. In addition to the expressions in the standard Up-transform in Sec. 2.2 to determine $\delta\alpha^{\mathrm{LS}}$, δp^{LS}, $\delta\theta^{\mathrm{vLS}}$ and $\delta\mathbf{u}$, the following is also required

$$\delta\alpha^{\mathrm{LS}} = \alpha^{\mathrm{LS}^2}\delta\rho^{\mathrm{LS}}, \tag{4.27}$$

$$\delta\alpha^{\mathrm{LS}} = \mathbf{A}^{\rho}\delta\rho^{\mathrm{LS}}. \tag{4.27a}$$

The LS/CS scale separation would be maintained in $\delta\rho^{\mathrm{LS}}$ and $\delta\alpha^{\mathrm{LS}}$ only for constant α^{LS}. This is not constant but, as before, we put this issue to one side for now.

4.8 The Tp-transform for option B

Input fields	Output fields
δu, δv, δw, $\delta\theta$, δp and δq_{T}	$\delta\psi^{\mathrm{LS}}$, $\delta\chi^{\mathrm{u}}$, δp^{A}, $\delta\psi^{\mathrm{CS}}$, $\delta\theta^{\mathrm{vCS}}$, $\delta\mu$

This Tp-transform is designed to be the inverse of the Up-transform given below. To understand this Tp-transform, it is recommended that Sec. 4.9 is read first.

Tp.1. Calculate the streamfunction - see (2.4)

$$\delta\psi = \nabla_{\mathrm{h}}^{-2}\left\{\mathbf{k}\cdot\left[\nabla\times\begin{pmatrix}\delta u\\ \delta v\\ 0\end{pmatrix}\right]\right\}, \tag{4.28*}$$

$$\delta\psi = \mathbf{Y}\delta\mathbf{u}_{w=0}, \tag{4.28a}$$

where $\delta\mathbf{u}_{w=0}$ is a 3-D wind vector with zero vertical component.

Tp.2. Calculate the velocity potential - see (2.5)

$$\delta\chi = \nabla_{\mathrm{h}}^{-2}\left\{\nabla\cdot\begin{pmatrix}\delta u\\ \delta v\\ 0\end{pmatrix}\right\}, \tag{4.29}$$

$$\delta\chi = \mathbf{C}\delta\mathbf{u}_{w=0}. \tag{4.29a}$$

Tp.3. Calculate the balanced component of the velocity potential - see (2.6)

$$\delta\chi^{\mathrm{b}} = \mathbf{B}_{\mathrm{v}}^{\chi\psi}\mathbf{B}_{\mathrm{v}}^{\psi\psi^{-1}}\mathbf{S}_{\mathrm{G}}^{\mathrm{LS}}\delta\psi, \tag{4.30}$$

$$= \mathbf{X}\mathbf{S}_{\mathrm{G}}^{\mathrm{LS}}\delta\psi, \tag{4.30a}$$

where $\mathbf{S}_{\mathrm{G}}^{\mathrm{LS}}$ is a filter that allows through only scales where the geostrophic balance approximation is valid. At smaller scales, it is assumed that there is no balanced velocity potential.

Tp.4. Calculate the unbalanced component of the velocity potential - see (2.7)

$$\delta\chi^{\mathrm{u}} = \delta\chi - \delta\chi^{\mathrm{b}}. \tag{4.31* / 4.31a}$$

Tp.5. Calculate the level-by-level geostrophically balanced pressure - see (2.8)

$$\delta p^{\mathrm{G}} = \nabla_{\mathrm{h}}^{-2}\left\{\nabla_{\mathrm{h}}\cdot\left[f\rho\nabla_{\mathrm{h}}\mathbf{S}_{\mathrm{G}}^{\mathrm{LS}}\delta\psi\right]\right\}, \tag{4.32}$$

$$\delta p^{\mathrm{G}} = \mathbf{L}\mathbf{S}_{\mathrm{G}}^{\mathrm{LS}}\delta\psi. \tag{4.32a}$$

Note the presence of the large-scale filter, $\mathbf{S}_{\mathrm{G}}^{\mathrm{LS}}$, which is not present in the standard transforms.

Tp.6. Calculate the vertically regressed geostrophically balanced pressure - see (2.9)

$$\delta p^{\mathrm{F}} = \mathbf{B}_{\mathrm{v}}^{p^{\mathrm{F}}p^{\mathrm{G}}}\mathbf{B}_{\mathrm{v}}^{p^{\mathrm{G}}p^{\mathrm{G}-1}}\delta p^{\mathrm{G}}, \tag{4.33}$$

$$\delta p^{\mathrm{F}} = \mathbf{G}\delta p^{\mathrm{G}}. \tag{4.33a}$$

Tp.7. Calculate the virtual potential temperature - see (2.1)

$$\delta\theta^{\mathrm{v}} = [1 + (\varepsilon^{-1} - 1)\,q]\,\delta\theta + \theta\,(\varepsilon^{-1} - 1)\,\delta q_{\mathrm{T}}, \tag{4.34}$$

$$\delta\theta^{\mathrm{v}} = \Theta^{\theta}\delta\theta + \Theta^{q}\delta q_{\mathrm{T}}.$$

Tp.8. Calculate the moisture control variable - see (2.3)

$$\delta\mu = a\left(\frac{1}{q_{\mathrm{s}}}\delta q_{\mathrm{T}} - h_1\frac{q\Pi}{q_{\mathrm{s}}}\frac{d\,\ln e_{\mathrm{s}}}{dT}\delta\theta - h_2\frac{q}{pq_{\mathrm{s}}}\left[\kappa\Pi\theta\frac{d\,\ln e_{\mathrm{s}}}{dT} - 1\right]\delta p\right), \tag{4.35*}$$

$$\delta\mu = \mathbf{M}^{q}\delta q_{\mathrm{T}} + \mathbf{M}^{\theta}\delta\theta + \mathbf{M}^{p}\delta p. \tag{4.35a}$$

Tp.9. Calculate the large-scale pressure

The following result is a combination of steps in the Up-transform (it is recommended that Sec. 4.9 is read first) and is given here in operator form only. This result is derived in Sec. 4.10 after the Up-transform is presented

$$\delta p^{\mathrm{LS}} =$$

$$\left(\mathbf{P}^{\alpha}\mathbf{A}^{\rho}[\mathbf{R}^{p} + \mathbf{R}^{\theta}\mathbf{P}] + \mathbf{P}^{p} + \mathbf{P}^{\theta^{\mathrm{LS}}}\mathbf{P} - \mathbf{P}^{\theta^{\mathrm{CS}}}\mathbf{P} + \Pi\right)^{-1}\left(\Pi\delta p - \mathbf{P}^{\theta^{\mathrm{CS}}}\delta\theta^{\mathrm{v}} - \mathbf{P}^{u}\delta\mathbf{u}\right). \tag{4.36a}$$

Operators that have not been defined so far are defined in the course of Sec. (4.9).

Tp.10. Calculate the convective-scale pressure

$$\delta p^{\mathrm{CS}} = \delta p - \delta p^{\mathrm{LS}}. \qquad (4.37\,/\,4.37\mathrm{a})$$

Tp.11. Calculate the geostrophically unbalanced pressure

$$\delta p^{\mathrm{A}} = \delta p^{\mathrm{LS}} - \delta p^{\mathrm{F}}. \qquad (4.38^*\,/\,4.38\mathrm{a})$$

Tp.12. Calculate the large-scale virtual potential temperature - see (2.18)

$$\delta\theta^{\mathrm{v}^{\mathrm{LS}}} = \frac{\kappa g}{c_p}\left(\frac{\partial\Pi}{\partial z}\right)^{-2}\frac{\partial}{\partial z}\left(\frac{\Pi^{\mathrm{H}}\delta p^{\mathrm{LS}}}{p^{\mathrm{H}}}\right), \tag{4.39}$$

$$\delta\theta^{\mathrm{v}^{\mathrm{LS}}} = \mathbf{P}\delta p^{\mathrm{LS}}. \tag{4.39a}$$

Tp.13. Calculate the convective-scale virtual potential temperature

$$\delta\theta^{\mathrm{v}^{\mathrm{CS}}} = \delta\theta^{\mathrm{v}} - \delta\theta^{\mathrm{v}^{\mathrm{LS}}}. \qquad (4.40\,/\,4.40\mathrm{a})$$

4.9 The Up-transform for option B

Input fields	Output fields
$\delta\psi^{\mathrm{LS}}$, $\delta\chi^{\mathrm{u}}$, δp^{A}, $\delta\psi^{\mathrm{CS}}$, $\delta\theta^{\mathrm{v}^{\mathrm{CS}}}$, $\delta\mu$	δu, δv, δw, $\delta\theta$, δp $\delta\rho$, and δq_{T}

4. Option B: Allow for a non-hydrostatic potential temperature and non-hydrostatic pressure

Up.1. Calculate the balanced velocity potential (as Tp.3)

$$\delta\chi^{\mathrm{b}} = \mathbf{B}_{\mathrm{v}}^{\chi\psi}\mathbf{B}_{\mathrm{v}}^{\psi\psi^{-1}}\mathbf{S}_{\mathrm{G}}^{\mathrm{LS}}\delta\psi, \tag{4.41}$$

$$= \mathbf{X}\mathbf{S}_{\mathrm{G}}^{\mathrm{LS}}\delta\psi. \tag{4.41a}$$

Up.2. Calculate the velocity potential (as Tp.4)

$$\delta\chi = \delta\chi^{\mathrm{u}} + \delta\chi^{\mathrm{b}}. \tag{4.42 / 4.42a}$$

Up.3. Calculate the horizontal wind components (as Tp.1 and Tp.2)

$$\begin{pmatrix}\delta u\\ \delta v\end{pmatrix} = \nabla_{\mathrm{h}}\delta\chi + \mathbf{k}\times\nabla\delta\psi, \tag{4.43*}$$

$$\delta\mathbf{u}_2 = \mathbf{C}^{-1}\delta\chi + \mathbf{Y}^{-1}\delta\psi, \tag{4.43a}$$

where $\delta\mathbf{u}_2$ is a 2-D (horizontal) wind vector.

Up.4. Calculate the level-by-level geostrophically balanced pressure (as Tp.5)

$$\delta p^{\mathrm{G}} = \nabla_{\mathrm{h}}^{-2}\left\{\nabla_{\mathrm{h}}\cdot\left[f\rho\nabla_{\mathrm{h}}\mathbf{S}_{\mathrm{G}}^{\mathrm{LS}}\delta\psi\right]\right\}, \tag{4.44}$$

$$\delta p^{\mathrm{G}} = \mathbf{L}\mathbf{S}_{\mathrm{G}}^{\mathrm{LS}}\delta\psi. \tag{4.44a}$$

Up.5. Calculate the vertically regressed geostrophically balanced pressure (as Tp.6)

$$\delta p^{\mathrm{F}} = \mathbf{B}_{\mathrm{v}}^{p^{\mathrm{F}}p^{\mathrm{G}}}\mathbf{B}_{\mathrm{v}}^{p^{\mathrm{G}}p^{\mathrm{G}^{-1}}}\delta p^{\mathrm{G}}, \tag{4.45}$$

$$\delta p^{\mathrm{F}} = \mathbf{G}\delta p^{\mathrm{G}}. \tag{4.45a}$$

Up.6. Calculate the large-scale pressure (as Tp.11)

$$\delta p^{\mathrm{LS}} = \delta p^{\mathrm{A}} + \delta p^{\mathrm{F}}. \tag{4.46 / 4.46a}$$

Up.7. Calculate the large-scale virtual potential temperature (as Tp.12)

$$\delta\theta^{\mathrm{vLS}} = \frac{\kappa g}{c_p}\left(\frac{\partial\Pi}{\partial z}\right)^{-2}\frac{\partial}{\partial z}\left(\frac{\Pi^{\mathrm{H}}\delta p^{\mathrm{LS}}}{p^{\mathrm{H}}}\right), \tag{4.47}$$

$$\delta\theta^{\mathrm{vLS}} = \mathbf{P}\delta p^{\mathrm{LS}}, \tag{4.47a}$$

Up.8. Calculate the total virtual potential temperature (as Tp.13)

$$\delta\theta^{\mathrm{v}} = \delta\theta^{\mathrm{vCS}} + \delta\theta^{\mathrm{vLS}}. \tag{4.48 / 4.48a}$$

Up.9. Calculate the large-scale density component

$$\delta\rho^{\mathrm{LS}} = \rho\left(\frac{1-\kappa}{p}\delta p^{\mathrm{LS}} - \frac{1}{\theta^{\mathrm{v}}}\delta\theta^{\mathrm{vLS}}\right), \tag{4.49*}$$

$$\delta\rho^{\mathrm{LS}} = \mathbf{R}^{p}\delta p^{\mathrm{LS}} + \mathbf{R}^{\theta}\delta\theta^{\mathrm{vLS}}. \tag{4.49a}$$

Up.10. Calculate the large-scale specific mass component - see (4.27)

$$\delta\alpha^{\mathrm{LS}} = \alpha^{\mathrm{LS}^2}\delta\rho^{\mathrm{LS}}, \tag{4.50}$$

$$\delta\alpha^{\mathrm{LS}} = \mathbf{A}^{\rho}\delta\rho^{\mathrm{LS}}. \tag{4.50a}$$

Up.11. Calculate the vertical velocity component by solving either the incompressible equation

$$\frac{\partial\,(\delta w\rho_y)}{\partial z} + \frac{\partial\,(\delta\rho_y w)}{\partial z} = -\nabla_{\mathrm{h}}\cdot\left[\rho_y\begin{pmatrix}\delta u\\ \delta v\end{pmatrix}\right] - \nabla_{\mathrm{h}}\cdot\left[\delta\rho_y\begin{pmatrix}u\\ v\end{pmatrix}\right], \tag{4.51*}$$

$$\delta w \approx \mathbf{W}\delta\mathbf{u}_2, \tag{4.51a}$$

or by solving Richardson's equation (not shown). For simplicity, the matrix/vector version of (4.51) does not include the $\delta\rho$ contribution.

Up.12. Calculate the convective-scale pressure contribution by solving the anelastic balance equation (4.26/4.26a) (given here only in operator form). This step is associated with Tp.9.

$$\delta p^{\mathrm{CS}} = \Pi^{-1}\left(\mathbf{P}^{\alpha}\delta\alpha^{\mathrm{LS}} + \mathbf{P}^{p}\delta p^{\mathrm{LS}} + \mathbf{P}^{\theta^{\mathrm{LS}}}\delta\theta^{\mathrm{v}^{\mathrm{LS}}} + \mathbf{P}^{\theta^{\mathrm{CS}}}\delta\theta^{\mathrm{v}^{\mathrm{CS}}} + \mathbf{P}^{u}\delta\mathbf{u}\right), \tag{4.52a}$$

where $\delta\mathbf{u}$ is a 3-D wind field, the operators in (4.52a) appear in (4.26a) and the operator Π is

$$\Pi = \nabla^2 \bullet \quad + g\frac{c_v}{c_p}\frac{\partial}{\partial z}\left(\frac{1}{\alpha^{\mathrm{LS}}}\frac{\bullet}{p^{\mathrm{LS}}}\right). \tag{4.53}$$

Up.13. Calculate the total pressure (as Tp.10)

$$\delta p = \delta p^{\mathrm{LS}} + \delta p^{\mathrm{CS}}. \tag{4.54* / 4.54a}$$

Up.14. Calculate together the potential temperature and the total specific humidity (using the definition of virtual potential temperature, $\delta\theta^{\mathrm{v}} = [1 + (\varepsilon^{-1} - 1)\,q]\,\delta\theta + \theta\,(\varepsilon^{-1} - 1)\,\delta q_{\mathrm{T}}$ (used in Tp.7), and Tp.8)

$$\delta q_{\mathrm{T}} = \left\{ah_1\frac{q\Pi}{q_{\mathrm{s}}}\frac{d\,\ln e_{\mathrm{s}}}{dT}\delta\theta^{\mathrm{v}} + [1 + (\varepsilon^{-1} - 1)\,q]\,\delta\mu + [1 + (\varepsilon^{-1} - 1)\,q]\,ah_2\frac{q}{pq_{\mathrm{s}}}\left[\kappa\Pi\theta\frac{d\,\ln e_{\mathrm{s}}}{dT} - 1\right]\delta p\right\} \Big/ \left\{ah_1\frac{q\Pi}{q_{\mathrm{s}}}\frac{d\,\ln e_{\mathrm{s}}}{dT}\theta\,(\varepsilon^{-1} - 1) + [1 + (\varepsilon^{-1} - 1)\,q]\,\frac{a}{q_{\mathrm{s}}}\right\}, \tag{4.55*}$$

$$\delta q_{\mathrm{T}} = \mathbf{A}^{-1}\left\{\Theta^{\theta}\delta\mu - \mathbf{M}^{\theta}\delta\theta^{\mathrm{v}} - \Theta^{\theta}\mathbf{M}^{p}\delta p\right\}, \tag{4.55a}$$

$$\delta\theta = \left\{\frac{a}{q_{\mathrm{s}}}\delta\theta^{\mathrm{v}} - \theta\,(\varepsilon^{-1} - 1)\,\delta\mu - \theta\,(\varepsilon^{-1} - 1)\,ah_2\frac{q}{pq_{\mathrm{s}}}\left[\kappa\Pi\theta\frac{d\,\ln e_{\mathrm{s}}}{dT} - 1\right]\delta p\right\} \Big/ \left\{ah_1\frac{q\Pi}{q_{\mathrm{s}}}\frac{d\,\ln e_{\mathrm{s}}}{dT}\theta\,(\varepsilon^{-1} - 1) + [1 + (\varepsilon^{-1} - 1)\,q]\,\frac{a}{q_{\mathrm{s}}}\right\}, \tag{4.56*}$$

$$\delta\theta = \mathbf{A}^{-1}\left\{\Theta^{q}\delta\mu - \mathbf{M}^{q}\delta\theta^{\mathrm{v}} - \Theta^{q}\mathbf{M}^{p}\delta p\right\}. \tag{4.56a}$$

4.10 Derivation of equation (4.36a)

Equation (4.36a) is a diagnostic equation giving the large-scale pressure for a given total pressure and virtual potential temperature. It is derived by combining some of the equations presented as part of the Up-transform in Sec. 4.9 as the prescription below. The derivation is made in operator notation at first (for conciseness) but the result is translated to explicit form at the end. This derivation is straightforward and involves only simple substitution of variables from Sec. 4.9.

The starting point is (4.52a), which is the diagnostic equation for convective-scale pressure. The strategy is to substitute all increment states that appear in this equation with either increments of large-scale pressure (the unknown) or increments that are known before the Tp.9 stage. First eliminate $\delta\alpha^{\mathrm{LS}}$ with (4.50a) and eliminate $\delta\theta^{\mathrm{vCS}}$ with (4.48a)

$$\begin{aligned}\delta p^{\mathrm{CS}} &= \Pi^{-1}\left(\mathbf{P}^{\alpha}\delta\alpha^{\mathrm{LS}} + \mathbf{P}^{p}\delta p^{\mathrm{LS}} + \mathbf{P}^{\theta^{\mathrm{LS}}}\delta\theta^{\mathrm{v}^{\mathrm{LS}}} + \mathbf{P}^{\theta^{\mathrm{CS}}}\delta\theta^{\mathrm{v}^{\mathrm{CS}}} + \mathbf{P}^{u}\delta\mathbf{u}\right),\\ &= \Pi^{-1}\left(\mathbf{P}^{\alpha}\mathbf{A}^{\rho}\delta\rho^{\mathrm{LS}} + \mathbf{P}^{p}\delta p^{\mathrm{LS}} + \mathbf{P}^{\theta^{\mathrm{LS}}}\delta\theta^{\mathrm{v}^{\mathrm{LS}}} + \mathbf{P}^{\theta^{\mathrm{CS}}}(\delta\theta^{\mathrm{v}} - \delta\theta^{\mathrm{v}^{\mathrm{LS}}}) + \mathbf{P}^{u}\delta\mathbf{u}\right).\end{aligned}$$

Next eliminate $\delta\rho^{\mathrm{LS}}$ with (4.49a) and factorise

$$\begin{aligned}\delta p^{\mathrm{CS}} &= \Pi^{-1}\left(\mathbf{P}^{\alpha}\mathbf{A}^{\rho}(\mathbf{R}^{p}\delta p^{\mathrm{LS}} + \mathbf{R}^{\theta}\delta\theta^{\mathrm{v}^{\mathrm{LS}}}) + \mathbf{P}^{p}\delta p^{\mathrm{LS}} + \mathbf{P}^{\theta^{\mathrm{LS}}}\delta\theta^{\mathrm{v}^{\mathrm{LS}}} + \mathbf{P}^{\theta^{\mathrm{CS}}}(\delta\theta^{\mathrm{v}} - \delta\theta^{\mathrm{v}^{\mathrm{LS}}}) + \mathbf{P}^{u}\delta\mathbf{u}\right),\\ &= \Pi^{-1}\left((\mathbf{P}^{\alpha}\mathbf{A}^{\rho}\mathbf{R}^{p} + \mathbf{P}^{p})\delta p^{\mathrm{LS}} + (\mathbf{P}^{\alpha}\mathbf{A}^{\rho}\mathbf{R}^{\theta} + \mathbf{P}^{\theta^{\mathrm{LS}}} - \mathbf{P}^{\theta^{\mathrm{CS}}})\delta\theta^{\mathrm{v}^{\mathrm{LS}}} + \mathbf{P}^{\theta^{\mathrm{CS}}}\delta\theta^{\mathrm{v}} + \mathbf{P}^{u}\delta\mathbf{u}\right).\end{aligned}$$

Next eliminate $\delta\theta^{\mathrm{v}^{\mathrm{LS}}}$ with (4.47a) and factorise

$$\begin{aligned}\delta p^{\mathrm{CS}} &= \Pi^{-1}\left((\mathbf{P}^{\alpha}\mathbf{A}^{\rho}\mathbf{R}^{p} + \mathbf{P}^{p})\delta p^{\mathrm{LS}} + (\mathbf{P}^{\alpha}\mathbf{A}^{\rho}\mathbf{R}^{\theta} + \mathbf{P}^{\theta^{\mathrm{LS}}} - \mathbf{P}^{\theta^{\mathrm{CS}}})\mathbf{P}\delta p^{\mathrm{LS}} + \mathbf{P}^{\theta^{\mathrm{CS}}}\delta\theta^{\mathrm{v}} + \mathbf{P}^{u}\delta\mathbf{u}\right),\\ &= \Pi^{-1}\left((\mathbf{P}^{\alpha}\mathbf{A}^{\rho}\mathbf{R}^{p} + \mathbf{P}^{p} + \mathbf{P}^{\alpha}\mathbf{A}^{\rho}\mathbf{R}^{\theta}\mathbf{P} + \mathbf{P}^{\theta^{\mathrm{LS}}}\mathbf{P} - \mathbf{P}^{\theta^{\mathrm{CS}}}\mathbf{P})\delta p^{\mathrm{LS}} + \mathbf{P}^{\theta^{\mathrm{CS}}}\delta\theta^{\mathrm{v}} + \mathbf{P}^{u}\delta\mathbf{u}\right).\end{aligned}$$

The left-hand side can be rewritten with (4.54a)

$$\delta p - \delta p^{\mathrm{LS}} = \Pi^{-1}\left((\mathbf{P}^{\alpha}\mathbf{A}^{\rho}\mathbf{R}^{p} + \mathbf{P}^{p} + \mathbf{P}^{\alpha}\mathbf{A}^{\rho}\mathbf{R}^{\theta}\mathbf{P} + \mathbf{P}^{\theta^{\mathrm{LS}}}\mathbf{P} - \mathbf{P}^{\theta^{\mathrm{CS}}}\mathbf{P})\delta p^{\mathrm{LS}} + \mathbf{P}^{\theta^{\mathrm{CS}}}\delta\theta^{\mathrm{v}} + \mathbf{P}^{u}\delta\mathbf{u}\right),$$

which can be rearranged

$$\delta p - \Pi^{-1}\left(\mathbf{P}^{\theta^{\mathrm{CS}}}\delta\theta^{\mathrm{v}} - \mathbf{P}^{u}\delta\mathbf{u}\right) = \Pi^{-1}\left((\mathbf{P}^{\alpha}\mathbf{A}^{\rho}\mathbf{R}^{p} + \mathbf{P}^{p} + \mathbf{P}^{\alpha}\mathbf{A}^{\rho}\mathbf{R}^{\theta}\mathbf{P} + \mathbf{P}^{\theta^{\mathrm{LS}}}\mathbf{P} - \mathbf{P}^{\theta^{\mathrm{CS}}}\mathbf{P} + \Pi)\delta p^{\mathrm{LS}}\right).$$

Further rearranging gives

$$\begin{aligned}\Pi\delta p - \mathbf{P}^{\theta^{\mathrm{CS}}}\delta\theta^{\mathrm{v}} - \mathbf{P}^{u}\delta\mathbf{u} &= \left(\mathbf{P}^{\alpha}\mathbf{A}^{\rho}\mathbf{R}^{p} + \mathbf{P}^{p} + \mathbf{P}^{\alpha}\mathbf{A}^{\rho}\mathbf{R}^{\theta}\mathbf{P} + \mathbf{P}^{\theta^{\mathrm{LS}}}\mathbf{P} - \mathbf{P}^{\theta^{\mathrm{CS}}}\mathbf{P} + \Pi\right)\delta p^{\mathrm{LS}},\\ &= \left(\mathbf{P}^{\alpha}\mathbf{A}^{\rho}[\mathbf{R}^{p} + \mathbf{R}^{\theta}\mathbf{P}] + \mathbf{P}^{p} + \mathbf{P}^{\theta^{\mathrm{LS}}}\mathbf{P} - \mathbf{P}^{\theta^{\mathrm{CS}}}\mathbf{P} + \Pi\right)\delta p^{\mathrm{LS}}.\end{aligned}$$

The increment δp^{LS} is thus found by inverting the appropriate operators

$$\delta p^{\mathrm{LS}} = \left(\mathbf{P}^{\alpha}\mathbf{A}^{\rho}[\mathbf{R}^{p} + \mathbf{R}^{\theta}\mathbf{P}] + \mathbf{P}^{p} + \mathbf{P}^{\theta^{\mathrm{LS}}}\mathbf{P} - \mathbf{P}^{\theta^{\mathrm{CS}}}\mathbf{P} + \Pi\right)^{-1}\left(\Pi\delta p - \mathbf{P}^{\theta^{\mathrm{CS}}}\delta\theta^{\mathrm{v}} - \mathbf{P}^{u}\delta\mathbf{u}\right). \quad (4.57a)$$

Recall that the following operators have been defined earlier

$$\mathbf{P}^{\alpha} = g\frac{c_v}{c_p}\frac{\partial}{\partial z}\left(\frac{\bullet}{\alpha^{\mathrm{LS}2}}\frac{p^{\mathrm{CS}}}{p^{\mathrm{LS}}}\right) + \nabla\cdot\left(\frac{\bullet}{\alpha^{\mathrm{LS}2}}(\mathbf{u}\cdot\nabla)\mathbf{u}\right) - \mathbf{k}\cdot\nabla\times\left(f\frac{\bullet}{\alpha^{\mathrm{LS}2}}\mathbf{u}\right) - g\frac{\partial}{\partial z}\left(\frac{\bullet}{\alpha^{\mathrm{LS}2}}\frac{\theta^{\mathrm{vCS}}}{\theta^{\mathrm{vLS}}}\right),$$

$$\mathbf{P}^{p} = g\frac{c_v}{c_p}\frac{\partial}{\partial z}\left(\frac{1}{\alpha^{\mathrm{LS}}}\frac{p^{\mathrm{CS}}}{p^{\mathrm{LS}2}}\bullet\right) - \nabla_{\mathrm{h}}^2\bullet,$$

$$\mathbf{P}^{\theta^{\mathrm{LS}}} = -g\frac{\partial}{\partial z}\left(\frac{1}{\alpha^{\mathrm{LS}}}\frac{\theta^{\mathrm{vCS}}}{\theta^{\mathrm{vLS}2}}\bullet\right),$$

$$\mathbf{P}^{\theta^{\mathrm{CS}}} = g\frac{\partial}{\partial z}\left(\frac{1}{\alpha^{\mathrm{LS}}}\frac{\bullet}{\theta^{\mathrm{vLS}}}\right),$$

$$\mathbf{P}^{u} = -\nabla\cdot\left(\frac{1}{\alpha^{\mathrm{LS}}}(\bullet\cdot\nabla)\mathbf{u}\right) - \nabla\cdot\left(\frac{1}{\alpha^{\mathrm{LS}}}(\mathbf{u}\cdot\nabla)\bullet\right) + \mathbf{k}\cdot\nabla\times\left(\frac{f}{\alpha^{\mathrm{LS}}}\bullet\right),$$

$$\mathbf{A}^{\rho} = \alpha^{\mathrm{LS}^2},$$

$$\mathbf{R}^{p} = \frac{\rho(1-\kappa)}{p},$$

$$\mathbf{R}^{\theta} = -\frac{\rho}{\theta^{\mathrm{v}}},$$

$$\mathbf{P} = \frac{\kappa g}{c_p}\left(\frac{\partial\Pi}{\partial z}\right)^{-2}\frac{\partial}{\partial z}\left(\frac{\Pi^{\mathrm{H}}\bullet}{p^{\mathrm{H}}}\right),$$

$$\Pi = \nabla^2\bullet + g\frac{c_v}{c_p}\frac{\partial}{\partial z}\left(\frac{1}{\alpha^{\mathrm{LS}}}\frac{\bullet}{p^{\mathrm{LS}}}\right).$$

4.11 The implied background error covariance matrix for option B

We now examine the implied background error covariance matrix for option B. The expressions can become very complicated and so we examine the implied covariances for the variables $\delta\psi$, $\delta\chi$, δp and $\delta\theta^{\mathrm{v}}$ only. Consequently, we need to involve the control variables $\delta\psi$, $\delta\chi^{\mathrm{u}}$, δp^{A} and $\delta\theta^{\mathrm{vCS}}$ only. First, a summary of the Up-transform for this reduced variable set

$$\begin{pmatrix}\delta\psi\\ \delta\chi\\ \delta p\\ \delta\theta^{\mathrm{v}}\end{pmatrix} = \begin{pmatrix}\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{XS}_{\mathrm{G}}^{\mathrm{LS}} & \mathbf{I} & \mathbf{0} & \mathbf{0}\\ \Delta\mathbf{GLS}_{\mathrm{G}}^{\mathrm{LS}} + \Delta^{u} & \Gamma^{\chi} & \Delta & \mathbf{0}\\ \mathbf{PGLS}_{\mathrm{G}}^{\mathrm{LS}} & \mathbf{0} & \mathbf{P} & \mathbf{I}\end{pmatrix}\begin{pmatrix}\delta\psi\\ \delta\chi^{\mathrm{u}}\\ \delta p^{\mathrm{A}}\\ \delta\theta^{\mathrm{vCS}}\end{pmatrix}, \tag{4.58}$$

where

$$\Delta = \mathbf{I} + \Pi^{-1}\left[\mathbf{P}^{\alpha}\mathbf{A}^{\rho}(\mathbf{R}^{p} + \mathbf{R}^{\theta}\mathbf{P}) + \mathbf{P}^{p} + \mathbf{P}^{\theta^{\mathrm{LS}}}\mathbf{P}\right], \tag{4.59}$$

$$\Delta^{u} = \Pi^{-1}\mathbf{P}_2^{u}(\mathbf{C}^{-1}\mathbf{XS}_{\mathrm{G}}^{\mathrm{LS}} + \mathbf{Y}^{-1}), \tag{4.60}$$

$$\Gamma^{\chi} = \Pi^{-1}\mathbf{P}_2^{u}\mathbf{C}^{-1}, \tag{4.61}$$

This is considerably more complicated than the previous schemes. The implied covariances are

$$\mathbf{B}^{\mathrm{imp}} = \mathbf{U}_p\mathbf{B}_p\mathbf{U}_p^{\mathrm{T}}, \tag{4.62}$$

where $\mathbf{B}_p$ is the background error covariance matrix of the variables $\delta\psi$, $\delta\chi^{\mathrm{u}}$, δp^{A} and $\delta\theta^{\mathrm{vCS}}$ (which is block diagonal). This expands to

4.11 The implied background error covariance matrix for option B

$$
\mathbf{B}^{\mathrm{imp}} = \begin{pmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{X}\mathbf{S}_{\mathrm{G}}^{\mathrm{LS}} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \Delta\mathbf{G}\mathbf{L}\mathbf{S}_{\mathrm{G}}^{\mathrm{LS}} + \Delta^{u} & \Gamma^{\chi} & \Delta & \mathbf{0} \\ \mathbf{P}\mathbf{G}\mathbf{L}\mathbf{S}_{\mathrm{G}}^{\mathrm{LS}} & \mathbf{0} & \mathbf{P} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{B}_{p}^{\psi} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_{p}^{\chi^{\mathrm{u}}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{B}_{p}^{p^{\mathrm{A}}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{p}^{\theta^{\mathrm{v}}} \end{pmatrix} \times
$$

$$
\begin{pmatrix} \mathbf{I} & \mathbf{S}_{\mathrm{G}}^{\mathrm{LS}\mathrm{T}}\mathbf{X}^{\mathrm{T}} & \mathbf{S}_{\mathrm{G}}^{\mathrm{LS}\mathrm{T}}\mathbf{L}^{\mathrm{T}}\mathbf{G}^{\mathrm{T}}\Delta^{\mathrm{T}} + \Delta^{u\mathrm{T}} & \mathbf{S}_{\mathrm{G}}^{\mathrm{LS}\mathrm{T}}\mathbf{L}^{\mathrm{T}}\mathbf{G}^{\mathrm{T}}\mathbf{P}^{\mathrm{T}} \\ \mathbf{0} & \mathbf{I} & \Gamma^{\chi\mathrm{T}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \Delta^{\mathrm{T}} & \mathbf{P}^{\mathrm{T}} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{pmatrix},
$$

$$
= \begin{pmatrix}
\mathbf{B}_{p}^{\psi} & \mathbf{B}_{p}^{\psi}\mathbf{S}_{\mathrm{G}}^{\mathrm{LS}\mathrm{T}}\mathbf{X}^{\mathrm{T}} & \mathbf{B}_{p}^{\psi}(\mathbf{S}_{\mathrm{G}}^{\mathrm{LS}\mathrm{T}}\mathbf{L}^{\mathrm{T}}\mathbf{G}^{\mathrm{T}}\Delta^{\mathrm{T}} + \Delta^{u\mathrm{T}}) & \mathbf{B}_{p}^{\psi}\mathbf{S}_{\mathrm{G}}^{\mathrm{LS}\mathrm{T}}\mathbf{L}^{\mathrm{T}}\mathbf{G}^{\mathrm{T}}\mathbf{P}^{\mathrm{T}} \\
\mathbf{X}\mathbf{S}_{\mathrm{G}}^{\mathrm{LS}}\mathbf{B}_{p}^{\psi} & \mathbf{X}\mathbf{S}_{\mathrm{G}}^{\mathrm{LS}}\mathbf{B}_{p}^{\psi}\mathbf{S}_{\mathrm{G}}^{\mathrm{LS}\mathrm{T}}\mathbf{X}^{\mathrm{T}} + \mathbf{B}_{p}^{\chi^{\mathrm{u}}} & \mathbf{X}\mathbf{S}_{\mathrm{G}}^{\mathrm{LS}}\mathbf{B}_{p}^{\psi}(\mathbf{S}_{\mathrm{G}}^{\mathrm{LS}\mathrm{T}}\mathbf{L}^{\mathrm{T}}\mathbf{G}^{\mathrm{T}}\Delta^{\mathrm{T}} + \Delta^{u\mathrm{T}}) + \mathbf{B}_{p}^{\chi^{\mathrm{u}}}\Gamma^{\chi\mathrm{T}} & \mathbf{X}\mathbf{S}_{\mathrm{G}}^{\mathrm{LS}}\mathbf{B}_{p}^{\psi}\mathbf{S}_{\mathrm{G}}^{\mathrm{LS}\mathrm{T}}\mathbf{L}^{\mathrm{T}}\mathbf{G}^{\mathrm{T}}\mathbf{P}^{\mathrm{T}} \\
(\Delta\mathbf{G}\mathbf{L}\mathbf{S}_{\mathrm{G}}^{\mathrm{LS}} + \Delta^{u})\mathbf{B}_{p}^{\psi} & (\Delta\mathbf{G}\mathbf{L}\mathbf{S}_{\mathrm{G}}^{\mathrm{LS}} + \Delta^{u})\mathbf{B}_{p}^{\psi}\mathbf{S}_{\mathrm{G}}^{\mathrm{LS}\mathrm{T}}\mathbf{X}^{\mathrm{T}} + \Gamma^{\chi}\mathbf{B}_{p}^{\chi^{\mathrm{u}}} & (\Delta\mathbf{G}\mathbf{L}\mathbf{S}_{\mathrm{G}}^{\mathrm{LS}} + \Delta^{u})\mathbf{B}_{p}^{\psi}(\mathbf{S}_{\mathrm{G}}^{\mathrm{LS}\mathrm{T}}\mathbf{L}^{\mathrm{T}}\mathbf{G}^{\mathrm{T}}\Delta^{\mathrm{T}} + \Delta^{u\mathrm{T}}) + \Gamma^{\chi}\mathbf{B}_{p}^{\chi^{\mathrm{u}}}\Gamma^{\chi\mathrm{T}} + \Delta\mathbf{B}_{p}^{p^{\mathrm{A}}}\Delta^{\mathrm{T}} & (\Delta\mathbf{G}\mathbf{L}\mathbf{S}_{\mathrm{G}}^{\mathrm{LS}} + \Delta^{u})\mathbf{B}_{p}^{\psi}\mathbf{S}_{\mathrm{G}}^{\mathrm{LS}\mathrm{T}}\mathbf{L}^{\mathrm{T}}\mathbf{G}^{\mathrm{T}}\mathbf{P}^{\mathrm{T}} + \Delta\mathbf{B}_{p}^{p^{\mathrm{A}}}\mathbf{P}^{\mathrm{T}} \\
\mathbf{P}\mathbf{G}\mathbf{L}\mathbf{S}_{\mathrm{G}}^{\mathrm{LS}}\mathbf{B}_{p}^{\psi} & \mathbf{P}\mathbf{G}\mathbf{L}\mathbf{S}_{\mathrm{G}}^{\mathrm{LS}}\mathbf{B}_{p}^{\psi}\mathbf{S}_{\mathrm{G}}^{\mathrm{LS}\mathrm{T}}\mathbf{X}^{\mathrm{T}} & \mathbf{P}\mathbf{G}\mathbf{L}\mathbf{S}_{\mathrm{G}}^{\mathrm{LS}}\mathbf{B}_{p}^{\psi}(\mathbf{S}_{\mathrm{G}}^{\mathrm{LS}\mathrm{T}}\mathbf{L}^{\mathrm{T}}\mathbf{G}^{\mathrm{T}}\Delta^{\mathrm{T}} + \Delta^{u\mathrm{T}}) + \mathbf{P}\mathbf{B}_{p}^{p^{\mathrm{A}}}\Delta^{\mathrm{T}} & \mathbf{P}\mathbf{G}\mathbf{L}\mathbf{S}_{\mathrm{G}}^{\mathrm{LS}}\mathbf{B}_{p}^{\psi}\mathbf{S}_{\mathrm{G}}^{\mathrm{LS}\mathrm{T}}\mathbf{L}^{\mathrm{T}}\mathbf{G}^{\mathrm{T}}\mathbf{P}^{\mathrm{T}} + \mathbf{P}\mathbf{B}_{p}^{p^{\mathrm{A}}}\mathbf{P}^{\mathrm{T}} + \mathbf{B}_{p}^{\theta^{\mathrm{v}}}
\end{pmatrix}. \quad (4.63)
$$

This result can now be compared to (2.28) for the standard scheme and (3.25) for option A.

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