SOME ADJOINT PROPERTIES OF THE LAPLACIAN IN SPHERICAL CO-ORDINATES

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BACKGROUND

The Laplacian operator in spherical co-ordinates, and its transpose operator, arises frequently in global data assimilation problems. Here we discuss the following useful points:

1. The Laplacian in self-adjoint. This is proved here for the case of the Laplacian in spherical co-ordinates.
2. Given that the Laplacian in self-adjoint, it is straightforward to go on to compute its transpose. The transpose is sometimes a different quantity to the adjoint, even though the terms are often used interchangeably.

1. THE SELF-ADJOINT PROPERTY OF THE LAPLACIAN (PROVED IN SPHERICAL CO-ORDINATES)

If the ∇^2 operator is self adjoint, then it must satisfy the following relation,

$$\langle \vec{\alpha}, \nabla^2 \vec{\beta} \rangle = \langle \nabla^2 \vec{\alpha}, \vec{\beta} \rangle, \tag{1.1}$$

where $\vec{\alpha}$ and $\vec{\beta}$ are vector representations for the global functions $\alpha(\lambda, \phi)$ and $\beta(\lambda, \phi)$ respectively (where λ is longitude and ϕ is latitude). The bra-ket notation implies the following special inner product,

$$\langle \vec{\alpha}, \vec{\beta} \rangle \equiv \vec{\alpha}^T \mathbf{P} \vec{\beta}, \tag{1.2}$$

where $\mathbf{P}$ is a diagonal matrix of grid box areas (in the discrete version of the inner product) which are proportional to $\cos\phi$. This is consistent with the continuous version of Eq. (1.2) which is an integral over the sphere,

$$\langle \vec{\alpha}, \vec{\beta} \rangle = \int_{\lambda=0}^{2\pi} \mathrm{d}\lambda \int_{\phi=-\pi/2}^{\pi/2} \mathrm{d}\phi \; r^2 \cos\phi \alpha(\lambda, \phi) \beta(\lambda, \phi). \tag{1.3}$$

∇^2 in two-dimensional spherical co-ordinates (in the meteorology convention) is,

$$\nabla^2 \alpha = \frac{1}{r^2 \cos^2\phi} \frac{\partial^2 \alpha}{\partial \lambda^2} + \frac{1}{r^2 \cos\phi} \frac{\partial}{\partial \phi} \left(\cos\phi \frac{\partial \alpha}{\partial \phi} \right). \tag{1.4}$$

This means that for ∇^2 to be self-adjoint it must satisfy,

$$\int \mathrm{d}\lambda \int \mathrm{d}\phi \left[\alpha \frac{1}{\cos\phi} \frac{\partial^2 \beta}{\partial \lambda^2} + \alpha \frac{\partial}{\partial \phi} \left(\cos\phi \frac{\partial \beta}{\partial \phi} \right) \right] = \int \mathrm{d}\lambda \int \mathrm{d}\phi \left[\beta \frac{1}{\cos\phi} \frac{\partial^2 \alpha}{\partial \lambda^2} + \beta \frac{\partial}{\partial \phi} \left(\cos\phi \frac{\partial \alpha}{\partial \phi} \right) \right] \tag{1.5}$$

which has been constructed from Eqs. (1.1), (1.3) and (1.4), with the integration limits made implicit. The left hand side of Eq. (1.5) can be shown to be equal to the right hand side by doing integration-by-parts. This is shown for each term separately.

1a. Term 1 (with the longitude derivative)

Let, $\gamma = \partial\beta / \partial\lambda$, and perform a first application of integration-by-parts for the λ integral. The first term of the integral is first simplified,

$$\int d\lambda \int d\phi \alpha \frac{1}{\cos\phi}\frac{\partial^2 \beta}{\partial \lambda^2} = \int d\phi \frac{1}{\cos\phi}\left\{\int d\lambda\, \alpha \frac{\partial \gamma}{\partial \lambda}\right\}, \tag{1.6}$$

where the quantity inside the curly brackets will be integrated-by-parts. For later reference, the integration-by-parts formula is,

$$\int v\, du = [uv] - \int u\, dv. \tag{1.7}$$

From Eq. (1.6) we set the following,

$$\begin{matrix} v = \alpha & du = \partial\gamma/\partial\lambda\ d\lambda \\ dv = \partial\alpha/\partial\lambda\ d\lambda & u = \gamma \end{matrix}, \tag{1.8}$$

making the part of Eq. (1.6) inside the curly brackets into,

$$\begin{aligned} \left\{\int d\lambda\, \alpha \frac{\partial \gamma}{\partial \lambda}\right\} &= [\alpha\gamma]_{\lambda=0}^{2\pi} - \int d\lambda\, \gamma \frac{\partial \alpha}{\partial \lambda}, \\ &= -\int d\lambda\, \gamma \frac{\partial \alpha}{\partial \lambda}, \\ &= -\int d\lambda\, \frac{\partial \beta}{\partial \lambda}\frac{\partial \alpha}{\partial \lambda}. \end{aligned} \tag{1.9}$$

The first term of the first line (evaluated at the limits) is zero. The definition of γ used above has been used to give the last line. A second application of integration-by-parts, yields the following,

$$\begin{matrix} v = \partial\alpha/\partial\lambda & du = \partial\beta/\partial\lambda\ d\lambda \\ dv = \partial^2\alpha/\partial\lambda^2\ d\lambda & u = \beta \end{matrix},$$

$$\begin{aligned} \left\{\int d\lambda\, \alpha \frac{\partial \gamma}{\partial \lambda}\right\} &= -\left[\frac{\partial \alpha}{\partial \lambda}\beta\right]_{\lambda=0}^{2\pi} + \int d\lambda \frac{\partial^2 \alpha}{\partial \lambda^2}\beta, \\ &= \int d\lambda \frac{\partial^2 \alpha}{\partial \lambda^2}\beta, \end{aligned} \tag{1.10}$$

where, once more the first term of the first line (evaluated at the limits) is zero. The first term of the left-hand-side of Eq. (1.5) is then,

$$\int d\lambda \int d\phi\, \alpha \frac{1}{\cos\phi}\frac{\partial^2 \beta}{\partial \lambda^2} = \int d\lambda \int d\phi\, \beta \frac{1}{\cos\phi}\frac{\partial^2 \alpha}{\partial \lambda^2}, \tag{1.11}$$

which equals the first term on the right-hand-side.

1b. Term 2 (with the latitude derivative)

The second term in the integral on the left-hand-side of Eq. (1.5) is simplified with the new substitution, $\gamma = \cos\phi\ \partial\beta/\partial\phi$,

$$\int d\lambda \int d\phi\, \alpha \frac{\partial}{\partial \phi}\left(\cos\phi \frac{\partial \beta}{\partial \phi}\right) = \int d\lambda \left\{\int d\phi\, \alpha \frac{\partial \gamma}{\partial \phi}\right\}. \tag{1.12}$$

The term in curly brackets is to be manipulated. Applying the first phase of the integration-by-parts procedure (Eq. 1.7) to this with the following substitutions,

$$\begin{matrix} v = \alpha & du = \partial\gamma/\partial\phi\ d\phi \\ dv = \partial\alpha/\partial\phi\ d\phi & u = \gamma \end{matrix},$$

gives for the part of Eq. (1.12) inside the curly brackets,

$$\begin{aligned}\int \mathrm{d}\phi\, \alpha\frac{\partial\gamma}{\partial\phi} &= [\alpha\gamma]_{\phi=-\pi/2}^{\pi/2} - \int \mathrm{d}\phi\, \gamma\frac{\partial\alpha}{\partial\phi},\\ &= -\int \mathrm{d}\phi\, \gamma\frac{\partial\alpha}{\partial\phi},\\ &= -\int \mathrm{d}\phi\ \cos\phi\, \frac{\partial\beta}{\partial\phi}\frac{\partial\alpha}{\partial\phi}.\end{aligned} \tag{1.13}$$

The first term of the first line, evaluated at the limits is zero due to the presence of the $\cos\phi$ in the new definition of γ above. This definition of γ has been used to write Eq. (1.13). Applying the integration-by-parts formula once more yields the following,

$$\begin{array}{ll} v = \cos\phi\ \partial\alpha/\partial\phi & \mathrm{d}u = \partial\beta/\partial\phi\ \mathrm{d}\phi \\ \mathrm{d}v = \partial/\partial\phi(\cos\phi\ \partial\alpha/\partial\phi)\,\mathrm{d}\phi & u = \beta \end{array},$$

$$\begin{aligned}\int \mathrm{d}\phi\, \alpha\frac{\partial\gamma}{\partial\phi} &= -\left[\beta\cos\phi\,\frac{\partial\alpha}{\partial\phi}\right]_{\phi=-\pi/2}^{\pi/2} + \int \mathrm{d}\phi\,\beta\frac{\partial}{\partial\phi}\left(\cos\phi\,\frac{\partial\alpha}{\partial\phi}\right),\\ &= \int \mathrm{d}\phi\,\beta\frac{\partial}{\partial\phi}\left(\cos\phi\,\frac{\partial\alpha}{\partial\phi}\right),\end{aligned} \tag{1.14}$$

where, once more the presence of the $\cos\phi$ term means that the first term of the first line is zero evaluated at the limits. The second term of the left-hand-side of Eq. (1.5) is then,

$$\int \mathrm{d}\lambda \int \mathrm{d}\phi\, \alpha\frac{\partial}{\partial\phi}\left(\cos\phi\frac{\partial\beta}{\partial\phi}\right) = \int \mathrm{d}\lambda \int \mathrm{d}\phi\beta\frac{\partial}{\partial\phi}\left(\cos\phi\frac{\partial\alpha}{\partial\phi}\right). \tag{1.15}$$

Equations (1.11) and (1.15) together prove the self-adjoint property of the Laplacian operator.

2. THE TRANSPOSE OF THE LAPLACIAN OPERATOR

The inner product on the sphere, **P** (Eq. (1.2)), is not unity and so the transpose and adjoint of any operator defined in this space are not the same thing. Expanding-out Eq. (1.1) using the definition of Eq. (1.2) gives,

$$\begin{aligned}\vec{\alpha}^T\mathbf{P}\nabla^2\vec{\beta} &= \left(\nabla^2\vec{\alpha}\right)^T\mathbf{P}\vec{\beta},\\ &= \vec{\alpha}^T\left(\nabla^2\right)^T\mathbf{P}\vec{\beta}.\end{aligned} \tag{2.1}$$

Equation (2.1) leads to an expression for the transpose Laplacian,

$$\begin{aligned}\mathbf{P}\nabla^2 &= \left(\nabla^2\right)^T\mathbf{P},\\ \therefore\ \left(\nabla^2\right)^T &= \mathbf{P}\nabla^2\mathbf{P}^{-1},\end{aligned} \tag{2.2}$$

noting that **P** is diagonal and so is equal to its transpose. Thus the transpose Laplacian operator is not the same as the adjoint, but is very easy to compute.