How do observations affect background error covariance lengthscales?

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December 2005

Introduction

It is often reported that the correlation lengthscale of background errors is reduced in the presence of observations (Ingleby, 2001). This note discusses the mechanism of how this occurs. We look at the simpler problem of analysing how the presence of a single observation affects the correlation lengthscale of the analysis state, rather than the correlation lengthscale of the subsequent forecast (it is reasonable to expect that the qualitative characteristics of the analysis will be carried forward in the subsequent forecast).

Simple analysis

Consider a background error covariance matrix, **B**. Let it be homogeneous and isotropic, so that its representation in spectral space is diagonal (and depend upon total wavenumber only). Its spectral representation shall be used below. Let the observation system be denoted by the Jacobian, **H**, and let the error covariance of the observations be **R**. The error covariance matrix of the analysed state, **A**, is the inverse Hessian,

$$\mathbf{A} = (\mathbf{B}^{-1} + \mathbf{H}^T\mathbf{R}^{-1}\mathbf{H})^{-1}. \tag{1}$$

The space in which the analysis state, the **B**-matrix and the right-space of **H** is usually spatial. It is more convenient for this analysis to look at **A** in spectral space. Let the following operators be the forward and inverse Fourier transform operators,

$$\text{Forward FT:} \qquad \mathbf{F}^{-1}, \tag{2}$$

$$\text{Inverse FT:} \qquad \mathbf{F}. \tag{3}$$

The Fourier transform is orthogonal, $\mathbf{F}^T = \mathbf{F}^{-1}$. In one-dimension, the matrix elements of **F** are proportional to simple plane waves,

$$\mathbf{F}_{mq} = \frac{1}{\sqrt{N}} \exp ix_m k_q, \tag{4}$$

for N grid-points. The spectral-space version of Eq. (1) is $\mathbf{F}^T\mathbf{AF}$,

$$\begin{aligned}\mathbf{F}^T\mathbf{AF} &= \mathbf{F}^T(\mathbf{B}^{-1} + \mathbf{H}^T\mathbf{R}^{-1}\mathbf{H})^{-1}\mathbf{F},\\ &= (\mathbf{F}^T\mathbf{B}^{-1}\mathbf{F} + \mathbf{F}^T\mathbf{H}^T\mathbf{R}^{-1}\mathbf{HF})^{-1},\\ &= ([\mathbf{F}^T\mathbf{BF}]^{-1} + [\mathbf{HF}]^T\mathbf{R}^{-1}[\mathbf{HF}])^{-1}.\end{aligned} \tag{5}$$

The reason for converting to spectral space is for simplicity - we shall assume homogeneity throughout (and so the covariances in spectral space are diagonal) and we can infer lengthscales from the variance spectra - see below.

The background term in spectral space, $\mathbf{F}^T\mathbf{BF}$, is diagonal, with diagonal elements,

$$[\mathbf{F}^T\mathbf{BF}]_{qq} = \sigma_B^2(q). \tag{6}$$

Consider Eq. (5) with just one observation of grid-point l. The Jacobian is then,

$$\mathbf{H} = (0\,0\,0\,0\,1\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0), \tag{7}$$

with the '1' at position l. The operator $\mathbf{HF}$, which is the single row Jacobian acting on a spectral-space state, is made up of elements,

$$[\mathbf{HF}]_{1q} = \frac{1}{\sqrt{N}} \exp ix_l k_q. \tag{8}$$

By taking the transpose operator to include an additional complex conjugate operation, the operator $[\mathbf{HF}]^T \mathbf{R}^{-1} [\mathbf{HF}]$ is then,

$$[[\mathbf{HF}]^T \mathbf{R}^{-1} [\mathbf{HF}]]_{qk} = \frac{1}{\sqrt{N}} \exp -ix_l k_q \times \frac{1}{\sigma_{Ob}^2} \times \frac{1}{\sqrt{N}} \exp ix_l k_k,$$
$$= \frac{\exp ix_l (k_k - k_q)}{N\sigma_{Ob}^2}, \tag{9}$$

where σ_{Ob} is the observation standard deviation. Enforcing a homogeneous model, the contribution, $[\mathbf{HF}]^T \mathbf{R}^{-1} [\mathbf{HF}]$, is diagonal. Ignoring off-diagonal elements (setting them to zero), leaves the diagonal elements,

$$[[\mathbf{HF}]^T \mathbf{R}^{-1} [\mathbf{HF}]]_{qq} = \frac{1}{N\sigma_{Ob}^2}, \tag{10}$$

which is a constant. Wavenumber component q of Eq. (5) is thus,

$$[\mathbf{F}^T\mathbf{AF}]_{qq} = \left(\frac{1}{\sigma_B^2(q)} + \frac{1}{N\sigma_{Ob}^2}\right)^{-1},$$
$$= \left(\frac{N\sigma_{Ob}^2 + \sigma_B^2(q)}{\sigma_B^2(q) N\sigma_{Ob}^2}\right)^{-1} = \frac{\sigma_B^2(q) N\sigma_{Ob}^2}{N\sigma_{Ob}^2 + \sigma_B^2(q)} \tag{11}$$

Equation (11) says that those modes of the background state that have a small variance, ie $\sigma_B^2(q) << N\sigma_{Ob}^2$, will be unaffected by the observation. Those modes that have a large variance, ie $\sigma_B^2(q) >> N\sigma_{Ob}^2$, will have its variance reduced to a value, at most $N\sigma_{Ob}^2$. This is illustrated in Fig. 1. The variance of the longer modes (small wavenumbers) have reduced in value. As this has not affected the small modes (large wavenumbers), this has the effect to broaden the correlation spectra associated with the variance spectra in Fig. 1. This will shorten the lengthscale in positional space. In other words, the longer scales are analysed more than smaller scales, because they started with larger variance in **B**.

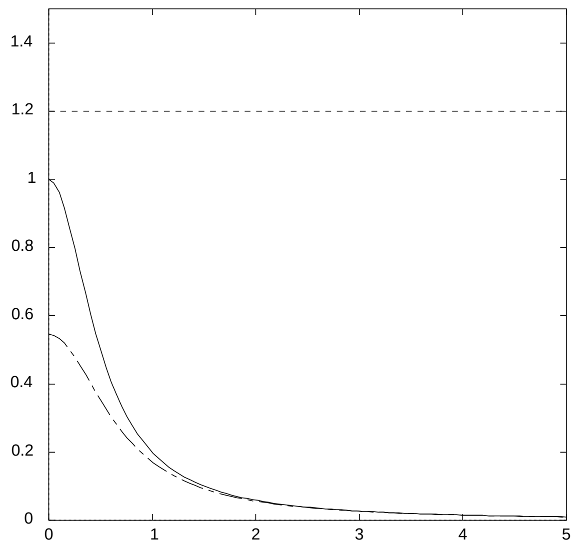

Figure 1: Variance spectra: **B**-matrix in spectral space (continuous curve), $N\sigma_{Ob}^2$ (straight dotted line) and **A**-matrix in spectral space (dash-dotted line).

Numerical study - dropping homogeneity

By dropping the homogeneity assumption, off-diagonal elements in Eq. (9) will be present. This is result in an **A**-matrix that has a reduced lengthscale, but only in the vicinity of the observation. This can be shown numerically.

Consider the one-dimensional system ($0 \leqslant x \leqslant 1$) with a number of observations near the middle of the domain. Let observation i make a direct measurement at position x. The row in the Jacobian will be zero apart from the elements corresponding to grid-points immediately before (x_1) and after (x_2) the observation. These will have elements,

$$1 - \frac{x - x_1}{x_2 - x_1} \qquad \text{and} \qquad \frac{x - x_1}{x_2 - x_1}, \tag{12}$$

by assuming linear interpolation.

In this numerical study, we need not invoke spectral space, and deal with small matrices directly. We use 30 points and 5 observations near the centre of the domain. The background error covariances have the simple form,

$$\mathbf{B} \; : \; \mathrm{COV}_B(\Delta x) = \frac{\sigma_B^2}{1 + (\Delta x / L)^2}, \tag{13}$$

where Δx is distance, $\sigma_B = 0.1$ and the correlation length parameter $L = 0.2$, and we invoke periodic boundary conditions for a well-behaved background error covariance matrix. The analysis error covariance matrix is then,

$$\mathbf{A} = (\mathbf{B}^{-1} + \mathbf{H}^T\mathbf{R}^{-1}\mathbf{H})^{-1}. \tag{14}$$

Plotted in Fig. 2 are the background error and analysis error correlation matrices,

$$\mathrm{COR}_B = \boldsymbol{\Sigma}_B^{-1}\mathbf{B}\boldsymbol{\Sigma}_B^{-1}, \tag{15}$$

$$\mathrm{COR}_A = \Sigma_A^{-1}\mathbf{A}\Sigma_A^{-1}, \qquad (16)$$

where Σ_B is the background error standard deviation matrix $\Sigma_B = \sigma_B\mathbf{I}$, and Σ_A is the analysis error standard deviation matrix.

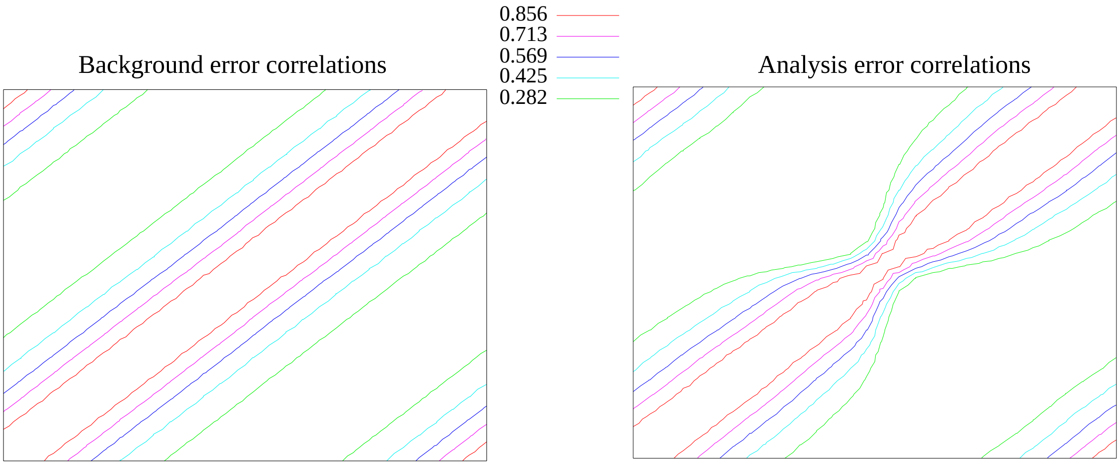

Figure 2: The background error correlations from Eqs. (13) and (15) (left) and the analysis error correlations from Eqs. (14) and (16) after the assimilation of five observations near the centre of the domain.

The analysis state shows clearly the reduction of correlation lengthscale in the analysis in the vicinity of the observation locations.

References

Ingleby N.B., 2001, The statistical structure of forecast errors and its representation in the Met Office global 3-dimensional variational data assimilation system, Quart. J. Roy. Met. Soc., 127, 209-231.