Multiple outer loops in Var without the T-transform

Ross Bannister, February 2004

Introductory comments

At present, the **T**-transform (the model-to-control space change of variables) is used at the start of each outer loop of Var. This is done to convert the background state (as a perturbation from a reference state) to control variable space allowing the background penalty and gradient in that space to be computed. For the *n*th outer loop, these quantities are,

$$J_B = \frac{1}{2}(\vec{v}_n{}' - \vec{v}_n'^B)^T(\vec{v}_n' - \vec{v}_n'^B), \tag{1}$$

$$\frac{\partial J_B}{\partial \vec{v}'} = (\vec{v}_n' - \vec{v}_n'^B). \tag{2}$$

If the reference state is the background state itself (as conventional in the first outer loop iteration) then the **T**-transform can be by-passed as the background increment is trivially zero in both model and control spaces. For subsequent outer loops, the reference state is no longer the background, but the 'analysis' found from the outer loop iteration immediately before. Thus if the reference state used for the *n*th outer loop is $\vec{x}_n^0$, then the reference state used for the next iteration is $\vec{x}_{n+1}^0$,

$$\vec{x}_{n+1}^0 = \vec{x}_n^0 + \mathbf{U}\vec{v}_n', \tag{3}$$

where **U** is the **U**-transform (**T** and **U** are approximate inverses of each other) and $\vec{v}_n'$ is the analysis increment in control space after loop *n*.

Currently, the background increment in control space on the $n+1$th iteration is found by computing,

$$\vec{v}_{n+1}'^B = \mathbf{T}(\vec{x}^B - \vec{x}_{n+1}^0), \tag{4}$$

which requires use of the (expensive) **T**-transform.

Dropping the T-transform

It is not necessary to use the **T**-transform for any outer loop if the above conventions are used. Equation (4) can be rearranged for $\vec{x}^B$,

$$\vec{x}^B = \mathbf{U}\vec{v}_{n+1}'^B + \vec{x}_{n+1}^0, \tag{5}$$

which is for the $n+1$th iteration. For the *n*th iteration there is a similar relation,

$$\vec{x}^B = \mathbf{U}\vec{v}_n'^B + \vec{x}_n^0. \tag{6}$$

Eliminating $\vec{x}^B$ from Eqs. (5) and (6) yields the recursive expression,

$$\vec{v}_{n+1}'^B = \vec{v}_n'^B + \mathbf{T}(\vec{x}_n^0 - \vec{x}_{n+1}^0), \tag{7}$$

where $\vec{x}_n^0 - \vec{x}_{n+1}^0$ can be eliminated from Eq. (3) giving,

$$\begin{aligned}\vec{v}_{n+1}'^B &= \vec{v}_n'^B - \mathbf{TU}\vec{v}_n', \\ &= \vec{v}_n'^B - \vec{v}_n'.\end{aligned} \tag{8}$$

Thus, it is simple to compute the background increment at outer loop $n+1$ if it is known at the previous loop (just subtract the control-space analysis from it, as in Eq. (8)). The background perturbation is always known at the previous loop because we know its value at the first loop (zero).