Linearized PV of the vertical normal modes in spectral space

Ross Bannister, May 2004, September 2004

1. INTRODUCTION

Potential vorticity (PV) appears in many different forms. It is often useful as a quantity that is conserved by the governing equations of motion, which forms a basis from which it can be derived. For each set of approximations to the equations of motion (e.g. quasi geostrophic), there is a different form of the PV.

In this document we find a form of PV derived from a different point of view. We derive PV as a quantity that is associated with the 'balanced' component of the flow in a set of linearized equations. For the 'PV control variable' project [1][2], it is necessary to derive a formula for PV in the space of spherical harmonics in the horizontal (spectral space) and in the space of normal modes in the vertical (where the linearized equations of motion are decoupled in the vertical). This gives an unusual form of PV, which is a departure from the usual forms expressed in physical space (longitude and latitude in the horizontal and height, pressure or isentropic co-ordinates in the vertical).

This document has the following structure. In section 2 we review some spectral space concepts necessary for this document, and in section 3 we use this information to derive the PV required.

2. SPECTRAL SPACE

To convert from physical space to spectral space, fields are projected onto the spherical harmonics. The equations of motion in spectral space govern how these projections evolve in time. Each spherical harmonic function is denoted by $Y_n^m(\lambda, \phi)$ - where λ is longitude, ϕ is latitude and m, n are integers that label the order of the spherical harmonics (see below). $Y_n^m(\lambda, \phi)$ comprises an associated Legendre polynomial in latitude, $P_n^m(\sin\phi)$ and a plane wave in longitude, $e^{im\lambda}$,

$$Y_n^m(\lambda, \phi) = P_n^m(\mu) e^{im\lambda}, \tag{2.1}$$

where $\mu = \sin\phi$. The spherical harmonics are useful on the sphere because they are eigenfunctions of the Laplacian, ∇^2 in spherical co-ordinates. We will not derive polynomials for $P_n^m(\mu)$, but we will state the orthogonality property of $Y_n^m(\lambda, \phi)$,

$$\int_{-1}^{1} \mathrm{d}\mu \int_{\lambda=0}^{2\pi} \mathrm{d}\lambda \, Y_n^{m^*}(\lambda, \phi) Y_{n'}^{m'}(\lambda, \phi) = 2\pi \delta_{nn'} \delta_{mm'}, \tag{2.2}$$

where the superscript * means complex conjugate and $\mathrm{d}\mu = \cos\phi \, \mathrm{d}\phi$. A spectral expansion of a field on the sphere, $u(\lambda, \phi)$ is denoted by the following linear combination of spherical harmonics,

$$u(\lambda, \phi) = \sum_{n=0}^{N} \sum_{m=-n}^{n} U_n^m Y_n^m(\lambda, \phi), \tag{2.3}$$

where U_n^m is the representation of the field in spectral space (the 'spectral coefficients'). The choice of truncation used in Eq. (2.3) is called 'triangular truncation' (Fig. 1). The spectral coefficients are found by projecting $u(\lambda, \phi)$ onto the spherical harmonics and by using the orthogonality property,

$$U_n^m = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} \mathrm{d}\phi \, \cos\phi \int_{\lambda=0}^{2\pi} \mathrm{d}\lambda \, Y_n^{m^*}(\lambda, \phi) u(\lambda, \phi). \tag{2.4}$$

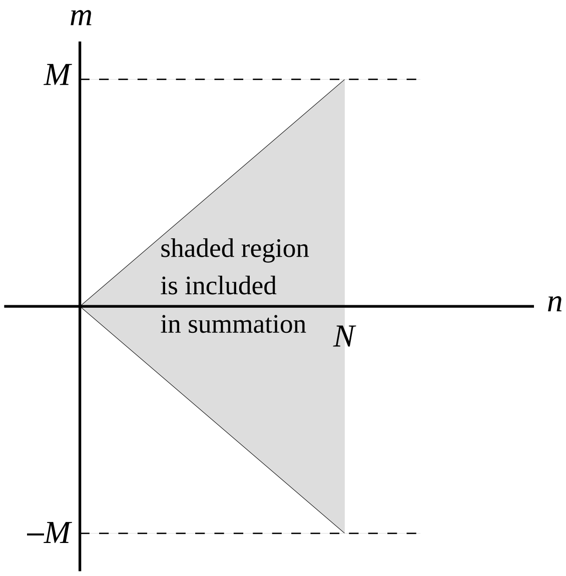

Figure 1: Triangular truncation used in the spectral expansion of Eq. (2.3).

3. LINEARIZED PV DERIVATION

There are a number of expressions that are needed to derive the expressions required for the PV derivation. Temperton [3] gives the vorticity and divergence equations in spectral space, which below have been linearized about a state of rest. Temperton's equations translate to,

$$\sigma\zeta_n^m = 2\Omega\left\{\frac{n+1}{n}\varepsilon_n^m\delta_{n-1}^m + \frac{n}{n+1}\varepsilon_{n+1}^m\delta_{n+1}^m - \frac{m}{n(n+1)}\zeta_n^m\right\}, \text{ and} \tag{3.1}$$

$$\sigma\delta_n^m = 2\Omega\left\{\frac{n+1}{n}\varepsilon_n^m\zeta_{n-1}^m + \frac{n}{n+1}\varepsilon_{n+1}^m\zeta_{n+1}^m - \frac{m}{n(n+1)}\delta_n^m\right\} - \frac{n(n+1)}{a^2}p_n^m. \tag{3.2}$$

Equation (3.1) is the vorticity equation and Eq. (3.2) is the divergence equation. Symbols are as follows: σ is angular frequency, ζ_n^m, δ_n^m and p_n^m are respectively the spectral vorticity, divergence and pressure perturbation associated with the spherical harmonic Y_n^m, $\mathbf{\Omega}$ is the rotation rate of the Earth, and ε_n^m is,

$$\varepsilon_n^m = \sqrt{\frac{n^2 - m^2}{4n^2 - 1}}. \tag{3.3}$$

In Eqs. (3.1) and (3.2), the prognostic variables differ slightly to those used by Temperton,

$$(\delta_n^m)_T = i\delta_n^m \exp(-i\sigma t), \tag{3.4}$$

$$(\zeta_n^m)_T = \zeta_n^m \exp(-i\sigma t), \quad \text{and} \tag{3.5}$$

$$(\phi_n^m)_T = -\frac{1}{\rho_0}p_n^m \exp(-i\sigma t)$$

where the subscript 'T' is the divergence used by Temperton and ρ_0 is the reference density. The factors of i make all terms real and the time functions, $\exp(-i\sigma t)$ remove the time derivatives of Temperton's equations.

We now write Eqs. (3.1) and (3.2) in vector/matrix notation. In this notation, the vector $\vec{\zeta}$ represents vorticity ζ_n^m ($\forall$ n, m and vertical level) and similarly for the other prognostic variables. Even in spectral space the vorticity and divergence equations remain coupled in the horizontal because of the Coriolis terms (inside the curly brackets in Eqs. (3.1) and (3.2)). Let the matrix **F** represent this coupling as in the first two terms of Eqs. (3.1) and (3.2) (including the $2\mathbf{\Omega}$ factor), let **G** be the diagonal matrix (again including the $2\mathbf{\Omega}$) that appears in the third term of Eqs. (3.1) and (3.2), and **H** be the diagonal matrix acting on pressure in Eq. (3.2). Equations (3.1) and (3.2) become,

$$\sigma\vec{\zeta} = \mathbf{F}\vec{\delta} - \mathbf{G}\vec{\zeta}, \tag{3.6}$$

$$\sigma\vec{\delta} = \mathbf{F}\vec{\zeta} - \mathbf{G}\vec{\delta} - \mathbf{H}\vec{p}. \tag{3.7}$$

In order to derive the vertical modes, we require the vertical structure equation. The vertical structure equation is found from a combination of the vertical momentum equation, the thermodynamic equation and the continuity equation. These equations are given by Thuburn [4] in a 'semi'-spectral space (semi-spectral space being the Fourier transformation in the zonal direction only; this gives fields as a function - in the horizontal - of zonal wavenumber and latitude). The three equations required for the vertical structure equation are then, from [4],

$$\sigma\rho_0\hat{w} + \left(\frac{\partial}{\partial z} + \frac{g}{c_s^2}\right)\hat{p} - \frac{\rho_0 g\hat{\theta}}{\theta_0} = 0, \tag{3.8}$$

$$-\sigma\frac{\rho_0 g\hat{\theta}}{\theta_0} + N_s^2\rho_0\hat{w} = 0, \quad \text{and} \tag{3.9}$$

$$-\sigma\hat{p} + c_s^2\rho_0\left\{\hat{\delta} + \left(\frac{\partial}{\partial z} + \frac{N_s^2}{g}\right)\hat{w}\right\} = 0, \tag{3.10}$$

where N_s^2 and c_s^2 are constants (see [4]) and ρ_0 and θ_0 are a reference profiles of density and potential temperature respectively. Equation (3.10) differs slightly from that given in [4] as we have identified that a group of terms involving horizontal winds as the divergence. The equations in [4] use semi-spectral space, but given that there are no latitudinal operators (like $\partial/\partial\phi$) and the coefficients are latitudinally independent, then the prognostic variables in Eqs. (3.8), (3.9) and (3.10) translate immediately to full spectral space,

$$\sigma\rho_0 w_n^m + \left(\frac{\partial}{\partial z} + \frac{g}{c_s^2}\right)p_n^m - \frac{\rho_0 g\theta_n^m}{\theta_0} = 0, \tag{3.11}$$

$$-\sigma\frac{\rho_0 g\theta_n^m}{\theta_0} + N_s^2\rho_0 w_n^m = 0, \quad \text{and} \tag{3.12}$$

$$-\sigma p_n^m + c_s^2\rho_0\left\{\delta_n^m + \left(\frac{\partial}{\partial z} + \frac{N_s^2}{g}\right)w_n^m\right\} = 0. \tag{3.13}$$

Use Eq. (3.12) to eliminate w_n^m in the other equations,

$$\left(\frac{\sigma^2}{N_s^2} - 1\right)\frac{\rho_0 g\theta_n^m}{\theta_0} + \left(\frac{\partial}{\partial z} + \frac{g}{c_s^2}\right)p_n^m = 0, \tag{3.14}$$

$$-\sigma p_n^m + c_s^2\left\{\rho_0\delta_n^m + \left(\frac{\partial}{\partial z} + \frac{N_s^2}{g}\right)\frac{\sigma}{N_s^2}\frac{\rho_0 g\theta_n^m}{\theta_0}\right\} = 0. \tag{3.15}$$

Now eliminate θ_n^m between Eqs. (3.14) and (3.15),

$$\rho_0\delta_n^m - \sigma\left[\frac{1}{c_s^2} + \left(\frac{\partial}{\partial z} + \frac{N_s^2}{g}\right)\left(\frac{1}{\sigma^2 - N_s^2}\right)\left(\frac{\partial}{\partial z} + \frac{g}{c_s^2}\right)\right]p_n^m = 0. \tag{3.16}$$

This equation is now written in vector/matrix notation as was done for Eqs. (3.6) and (3.7),

$$\rho_0\vec{\delta} - \sigma\left[c_s^{-2} + \mathbf{B}\right]\vec{p} = 0. \tag{3.17}$$

Equation (3.17) is to be used together with Eq. (3.6). Note that **B** in Eq. (3.7) is a vertical matrix that is independent of scale (m, n) and **F** in Eqs. (3.6) and (3.7) is a horizontal matrix that is independent of height. This means that matrices **B** and **F** commute. Eliminate $\vec{\delta}$ between Eqs. (3.6) and (3.17),

$$\sigma\left(\rho_0\vec{\zeta} - [c_s^{-2} + \mathbf{B}]\,\mathbf{F}\vec{p}\right) = -\mathbf{G}\rho_0\vec{\zeta}. \tag{3.18}$$

The fields are to be projected onto the vertical modes. These are the eigenvectors of the vertical operator **B**. Let vectors with a subscript 'B' represent the weights of vorticity and pressure of these vertical modes,

$$\rho_0\vec{\zeta} = \mathbf{E}\vec{\zeta}_{B,} \qquad \vec{p} = \mathbf{E}\vec{p}_B, \qquad (3.19)\ \ (3.20)$$

where **E** is the matrix of eigenvectors (columns). Substituting the above transform into Eq. (3.18) gives,

$$\sigma\left(\mathbf{E}\vec{\zeta}_B - [c_s^{-2} + \mathbf{B}]\,\mathbf{F}\mathbf{E}\vec{p}_B\right) = -\mathbf{G}\rho_0\mathbf{E}\vec{\zeta}_B. \tag{3.21}$$

Matrices **F** and **E** commute, as do **G** and **E**. Acting from the left with $\mathbf{E}^{-1}$ gives,

$$\sigma\left(\vec{\zeta}_B - [c_s^{-2} + \Lambda]\,\mathbf{F}\vec{p}_B\right) = -\mathbf{G}\vec{\zeta}_B \tag{3.22}$$

where $\Lambda$is the vertical structure matrix, **B**, projected onto its eigenvectors,

$$\Lambda = \mathbf{E}^{-1}\mathbf{B}\mathbf{E}. \tag{3.23}$$

In Eq. (3.22), **G** is regarded as a forcing term and so does not enter into the definition of PV. The PV of the normal modes is defined as the quantity whose evolution in this space does not depend upon divergence,

$$\vec{PV} = \vec{\zeta}_B - [c_s^{-2} + \Lambda]\,\mathbf{F}\vec{p}_B. \tag{3.24}$$

REFERENCES

[1] PV control variable web site at www.met.rdg.ac.uk/~ross/DARC/PVcv/PVcv.html

[2] Bannister R.N., The implementation of a PV-based leading control vector in variational data assimilation, available from the above web site.

[3] Temperton C., Implicit normal mode initialization for spectral models, Mon. Wea. Rev. 117, 436-451 (1989).

[4] Thuburn J., Wood N., Staniforth A., Normal modes of deep atmospheres. I: spherical geometry, Q. J. Roy. Met Soc. 128, 1771-1792 (2002).