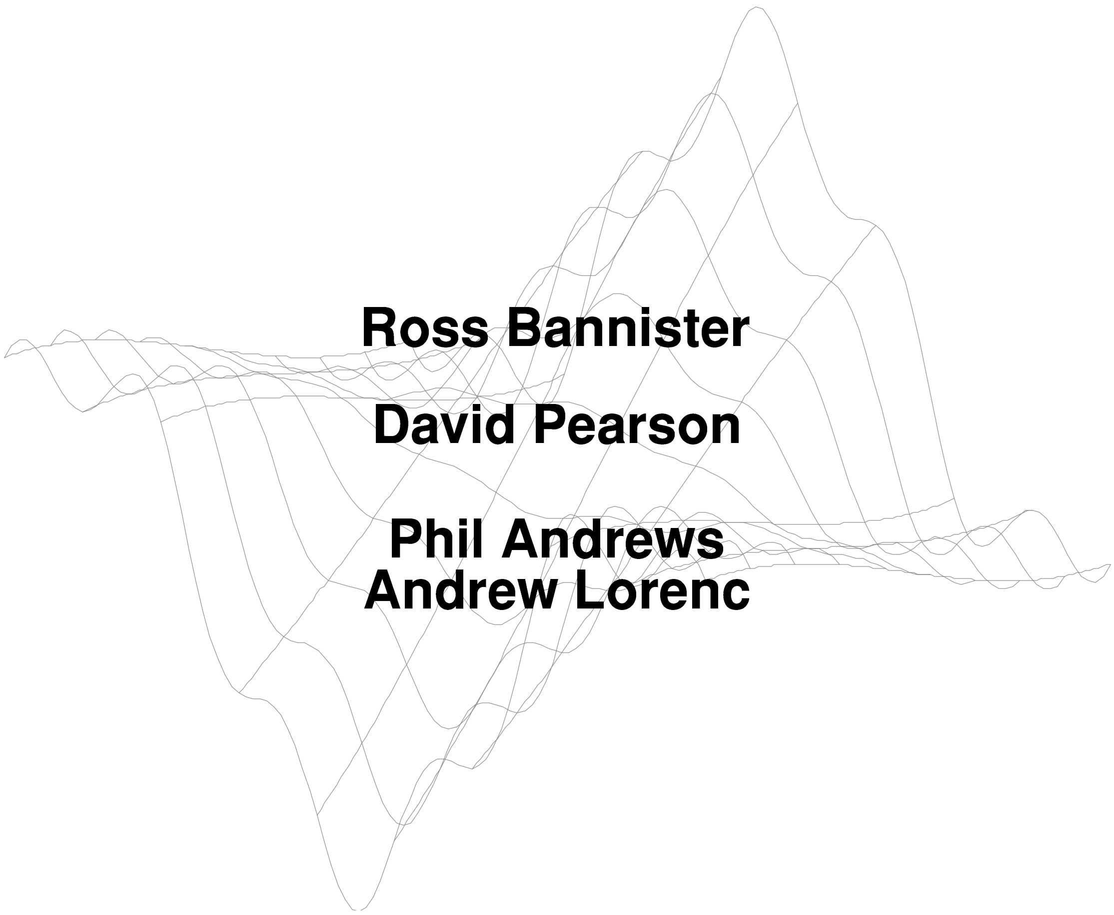
'Wavelets': what and why ?



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1. The background error covariance matrix (background background)

Bayes' theorem (Gaussian unbiased errors):

$$P(\boldsymbol{x} \backslash \boldsymbol{y}) \propto P(\boldsymbol{x}) \times P(\boldsymbol{y} \backslash \boldsymbol{x})$$

$$P(\boldsymbol{x}) = \frac{1}{2}(\boldsymbol{x}' - \boldsymbol{x}'^B)^T \mathbf{B}^{-1} (\boldsymbol{x}' - \boldsymbol{x}'^B)$$

$$P(\boldsymbol{y} \backslash \boldsymbol{x}) = \frac{1}{2}(\boldsymbol{y} - \boldsymbol{h}(\boldsymbol{x}))^T \mathbf{R}^{-1} (\boldsymbol{y} - \boldsymbol{h}(\boldsymbol{x}))$$

B is the error covariance matrix

The structure functions

- The columns (or rows) of **B**
- **B*x*** does a convolution by the structure functions
- Can be found in a single obs. test in Var.:

$$\text{analysis inc.} \quad \delta \boldsymbol{x} = \mathbf{K}(\boldsymbol{y} - \mathbf{H}\boldsymbol{x}^B)$$

$$\text{where} \quad \mathbf{K} = \mathbf{B}\mathbf{H}^T (\mathbf{H}\mathbf{B}\mathbf{H}^T + \mathbf{R})^{-1}$$

$$\text{single obs.} \quad \boldsymbol{y} = y, \ \mathbf{R} = \sigma^2$$

$$\text{then} \quad \mathbf{H} = (0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0) \quad (k\text{th element})$$

$$\Rightarrow \boldsymbol{y} - \mathbf{H}\boldsymbol{x}^B \equiv \Delta \boldsymbol{y} \equiv \Delta y$$

$$\Rightarrow \quad \mathbf{H}\mathbf{B}\mathbf{H}^T = B_{kk}$$

$$\Rightarrow \quad K_{l1} = B_{lk}(B_{kk} + \sigma^2)^{-1}$$

$$\therefore \quad \delta x_l = B_{lk} \times \frac{\Delta y}{B_{kk} + \sigma^2}$$

2. How is B computed?

A. Kalman filtering (1st choice)

$$\mathbf{B}(t) = \mathbf{M}^{t}_{t-\Delta t}\,\mathbf{P}_A(t-\Delta t)\,\mathbf{M}^{t\;T}_{t-\Delta t} + \mathbf{Q}(t-\Delta t)$$

$$\mathbf{P}_A(t-\Delta t) = (\mathbf{I} - \mathbf{K}(t-\Delta t)\,\mathbf{H}(t-\Delta t))\,\mathbf{B}(t-\Delta t)$$

Resolution	Matrix elements
$432 \times 325 \times 38 \times 4$ c.v.	4.6×10^{14}
$216 \times 163 \times 38 \times 4$ c.v.	2.9×10^{13}
$96 \times 73 \times 38 \times 4$ c.v.	1.1×10^{12}

B. Estimate covariance statistics fully (2nd choice)

$$\mathbf{B} = \langle \boldsymbol{x}' \boldsymbol{x}'^{T} \rangle$$

would like $\boldsymbol{x}' = \boldsymbol{x} - \boldsymbol{x}^{t}$

settle for $\boldsymbol{x}' \sim \boldsymbol{x}^{48} - \boldsymbol{x}^{24}$ (NMC e.g.)

C. Background error covariance modelling (third choice)

- Manageable number of parameters
- Capture essential features only
- Accept approximations
- Exploit symmetries

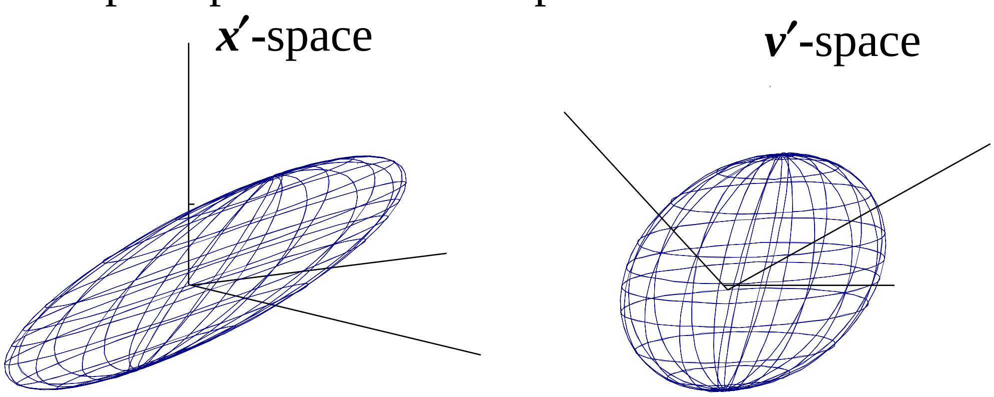
Assimilate using control parameters

$$\boldsymbol{x}' = \mathbf{U}\boldsymbol{v}'$$

- Control parameters are uncorrelated, $\mathbf{B}^{v} = \langle \boldsymbol{v}'\boldsymbol{v}'^{T} \rangle = \mathbf{I}$
- Covariance model contained inside transformation, $\mathbf{U}$

$$\mathbf{B} = \mathbf{U}\mathbf{U}^{T}$$

- Also helps to precondition the problem



- Assume $\mathbf{U}$ has form

$$\mathbf{U} = \mathbf{U}_p\mathbf{U}_v\mathbf{U}_h$$

$$\mathbf{T} = \mathbf{U}^{-1} = \mathbf{T}_h\mathbf{T}_v\mathbf{T}_p$$

$$(\lambda, \phi, z) \underset{\mathbf{U}_v}{\overset{\mathbf{T}_v}{\rightleftarrows}} (\lambda, \phi, \nu) \qquad (\lambda, \phi, \nu) \underset{\mathbf{U}_h}{\overset{\mathbf{T}_h}{\rightleftarrows}} (l, m, \nu)$$

3. Implied covariances (for a given parameter)

$$"\mathbf{B} = \mathbf{U}\mathbf{U}^T"$$

$$\mathbf{B}_p = \mathbf{U}_v\mathbf{U}_h\mathbf{U}_h^T\mathbf{U}_v^T$$

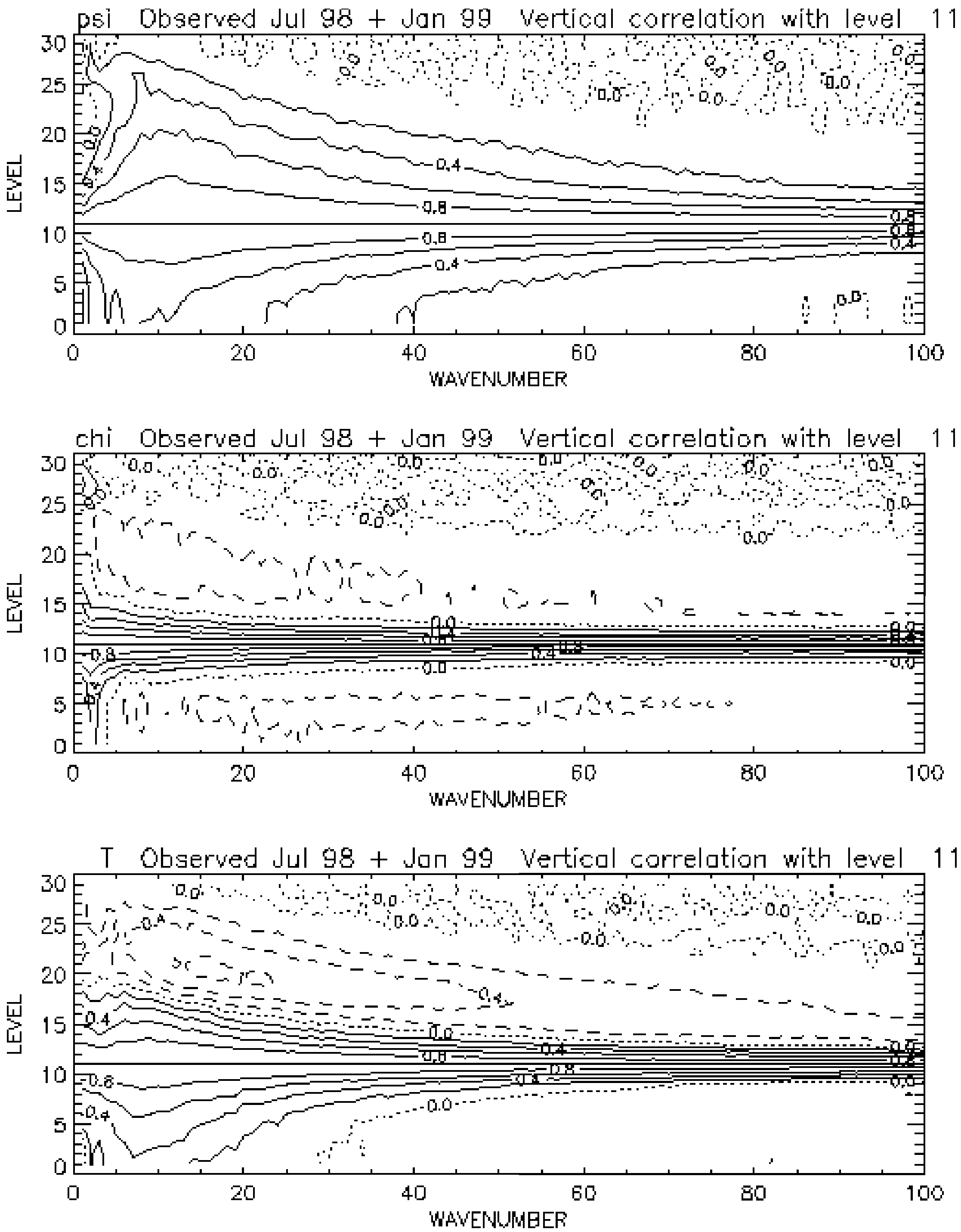
	T-transform	**U**-transform	$\mathbf{U}^T$-transform
VERTICAL	$\mathbf{M}^{-1/2}(\phi)\,\mathbf{Z}$	$\mathbf{Z}^{-1}\mathbf{M}^{1/2}(\phi)$	$\mathbf{M}^{1/2}(\phi)\,\mathbf{Z}^{-T}$
HORIZONTAL	$\Lambda_h^{-1/2}\mathbf{F}_h^{-1}$	$\mathbf{F}_h\Lambda_h^{1/2}$	$\Lambda_h^{1/2}\mathbf{F}_h^T$

General assumptions/remarks:

- Climatological error statistics used synoptically
- Errors in parameters are uncorrelated
- Horizontal structure functions are homogeneous and isotropic
- Horizontal length scales independent of height
- Order of transformations:
  - vertical covs. = $f(\phi)$, but $\neq f(l, m)$
  - horizontal covs. = $f(\nu)$, but $\neq f(z)$ (explicitly)
- Implied structure functions are *nearly* separable
  - $\mathbf{B}_p\delta(\lambda-\lambda_0, \phi-\phi_0, z-z_0) \approx f(\Delta r)\,g(\Delta z)$
  - (exactly separable if $\mathbf{U}_v$ independent of horiz. pos.)
  - Problems in capturing tropopause height and baroclinic structures

4. Real structure functions are not separable

Horizontal scales are associated with corresponding vertical scales



(from Ingleby, 2001)

Further evidence ...

Horizontal correlation length for T is shorter than that for $\mathbf{\Phi}$

$$\Delta_h^T < \Delta_h^{\Phi}$$

TABLE 1. DIFFERENTIAL LENGTH SCALES (KM) FOR FORECAST DIFFERENCES SELECTED MODEL LEVELS.

level	P	ψ	RKE	χ	DKE	Ap	RH	p	T
28	30	1353	366	1879	418	977		893	756
25	102	1057	277	1343	282	611	234	683	409
19	251	670	273	983	222	490	207	541	297
11	519	599	229	827	179	495	174	500	282
4	867	628	217	1045	195	536	171	499	251
2	961	581	199	724	181	433	164	466	211

(from Ingleby, 2001)

If short (long) horizontal lengths are associated with short (long) vertical lengths then,

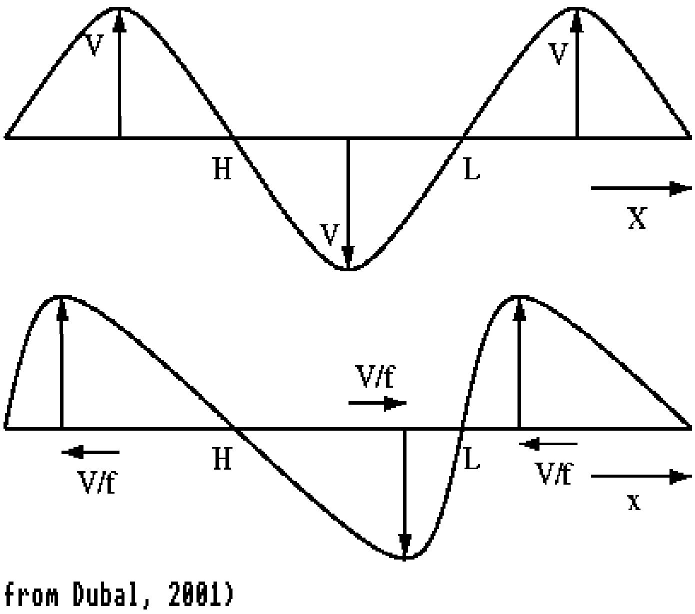
$$\Delta_v^T < \Delta_v^{\Phi}$$

which is true under hydrostatic balance,

$$\frac{\mathrm{d}\,\mathbf{\Phi}}{\mathrm{d}\,\ln p} = -\frac{R}{g}T$$

5. Innovative 'quick' fixes

- Use alternative parameters
  - E.g. PV/anti-PV to improve representation of balance in parameter transform (Cullen/Payne/Wlasak/Roulstone/Bannister)
- Geostrophic co-ordinate transform
  - Realistic treatment of fronts
  - $x_G = x_R + v_g / f, \; y_G = y_R + u_g / f$
  - $\mathbf{T} \rightarrow \mathbf{T}_h \mathbf{T}_v \mathbf{T}_g \mathbf{T}_p$



  - Errors of the day
    - Extra fit to 'bred' modes
    - Extra 'alpha' control variables, $\tilde{\boldsymbol{v}} = (\boldsymbol{v}, \alpha)$
    - $J \rightarrow J + J_\alpha, \; J_\alpha = ½ \, \alpha^T \mathbf{E}^{-1} \alpha$
  - Replacement for vertical\horizontal covariance model
    - 'WAVELET' (actually 'WAVEBAND') TRANSFORM

6. A waveband summation (WS) transform

The transform

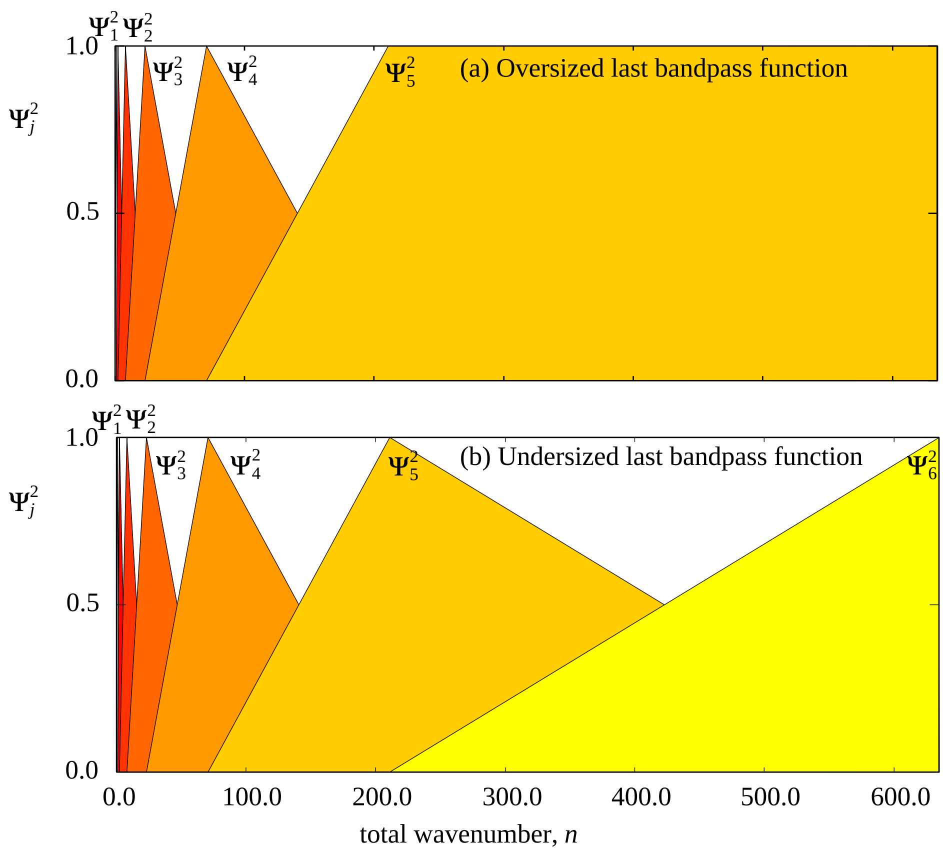
Current **U**-transform	'New' WS **U**-transform
$\mathbf{U}_p\mathbf{U}_v\mathbf{U}_h$	$\mathbf{U}_p \sum_{j=0}^{J} \mathbf{U}_{vj}\mathbf{U}_h\Psi_j^2$

Vertical transform has position and scale dependence

$$\mathbf{U}_p \qquad \sum_{j=0}^{J} \mathbf{U}_{vj} \qquad \mathbf{U}_h \qquad \Psi_j^2$$

$$\begin{matrix} u(\lambda, \phi, z) & & \psi(\lambda, \phi, z) & & \psi(\lambda, \phi, \nu) & & \psi(l, m, \nu) & & \psi(l, m, \nu) \\ v(\lambda, \phi, z) & \leftarrow & \chi(\lambda, \phi, z) & \leftarrow & \chi(\lambda, \phi, \nu) & \leftarrow & \chi(l, m, \nu) & \leftarrow & \chi(l, m, \nu) \\ \theta(\lambda, \phi, z) & & {}^{A}p(\lambda, \phi, z) & & {}^{A}p(\lambda, \phi, \nu) & & {}^{A}p(l, m, \nu) & & {}^{A}p(l, m, \nu) \end{matrix}$$

The wavebands



$$\sum_{j=0}^{J} \Psi_j^2 = 1$$

Are structure functions in the current transform separable?

$$\mathbf{B} = \mathbf{U}\mathbf{U}^T$$

$$\boldsymbol{x}' = \mathbf{U}\boldsymbol{v}' = \mathbf{U}_v\mathbf{U}_h\boldsymbol{v}' = \mathbf{Z}^{-1}\mathbf{M}^{1/2}\ \mathbf{F}_h\boldsymbol{\Lambda}_h^{1/2}\ \boldsymbol{v}'$$

$$x'(r, z) = \int \mathrm{d}v Z^{-1}(z, v) M^{1/2}(r, v) \int \mathrm{d}k F_h(r, k) \Lambda_h^{1/2}(k)\ v'(k, v)$$

$$x'(r', z') = \int \mathrm{d}v Z^{-1}(z', v) M^{1/2}(r', v) \int \mathrm{d}k F_h(r', k) \Lambda_h^{1/2}(k) \times$$

$$\int \mathrm{d}r\ \Lambda_h^{1/2}(k) F_h^{*}(r, k) \int \mathrm{d}z M^{1/2}(r, v) Z^{-1}(z, v)\ \ x(r, z)$$

$$\text{Let } x(r, z) = \delta(r - r'', z - z'')$$

$$x'(r', z') = \int \mathrm{d}v Z^{-1}(z', v) M^{1/2}(r', v) \int \mathrm{d}k F_h(r', k) \Lambda_h^{1/2}(k) \times$$

$$\Lambda_h^{1/2}(k) F_h^{*}(r'', k) M^{1/2}(r'', v) Z^{-1}(z'', v)$$

$$= \int \mathrm{d}v Z^{-1}(z', v) M^{1/2}(r', v) M^{1/2}(r'', v) Z^{-1}(z'', v) \times$$

$$\int \mathrm{d}k F_h(r', k) \Lambda_h^{1/2}(k) \Lambda_h^{1/2}(k) F_h^{*}(r'', k)$$

$$\text{If } M^{1/2}(r, v) \rightarrow M^{1/2}(v),$$

$$x'(r', z') = \int \mathrm{d}v Z^{-1}(z', v) M(v) Z^{-1}(z'', v) \times \int \mathrm{d}k F_h(r', k) \Lambda_h(k) F_h^{*}(r'', k)$$

$$= f(z') \times g(r')$$

The WS scheme?

$$\mathbf{U}_p \sum_{j=0}^{J} \mathbf{U}_{vj}\mathbf{U}_h\boldsymbol{\Psi}_j^2$$

$$x'(r', z') = \sum_{j, j'=0}^{J} \int \mathrm{d}v Z^{-1}(z', v) M_{j'}(v) Z^{-1}(z'', v) \times$$

$$\int \mathrm{d}k F_h(r', k) \Lambda_h(k) \Psi_{j'}^2(n) \Psi_j^2(n) F_h^{*}(r'', k)$$

Remarks on the waveband summation (WS) transform

- WS is the simplest of many proposed transforms
  - Some others require multiple control vectors
  - (current length $\times$ No. of bands)
- This WS transform reduces to current scheme if band index dropped
- Implied structure functions are not separable
  - (even if $\mathbf{U}_v$ independent of horiz. pos.)
- The Ψ_j^2 are not mutually orthogonal
  - No exact inverse transform
  - Need approximations when calibrating
- Increase scale resolution, decrease position resolution
- Other documents
  - Andrews P., *Covariance development strategy: a three point plan ...* (Var. working paper 20)
  - Ingleby N.B., *Options for improving and modelling background errors ...*
  - Andrews P., Lorenc A., *On introducing a multi-scale waveband summation covariance model ...* (Var. working paper 19)
  - Bannister R., *...a description of the proposed waveband summation transformation* (available via www.met.rdg.ac.uk/~ross/DARC/DataAssim.html)