'Balance' and Ensemble Localization

R.N. Bannister[1], L.H. Baker[1], S. Migliorini[1,2], A.C. Rudd[1,3]
[1]University of Reading, UK; [2]ECMWF, Reading, UK; [3]now CEH, Wallingford, UK

Forecast error covariance statistics, $\mathbf{P}^{\mathrm{f}}$, are prominent in the Kalman Filter, e.g.

$$\mathbf{x}^{\mathrm{a}} = \mathbf{x}^{\mathrm{b}} + \mathbf{P}^{\mathrm{f}}\mathbf{H}^{\mathrm{T}}\left(\mathbf{R} + \mathbf{H}\mathbf{P}^{\mathrm{f}}\mathbf{H}^{\mathrm{T}}\right)^{-1}\left(\mathbf{y} - \mathbf{H}\mathbf{x}^{\mathrm{b}}\right)$$

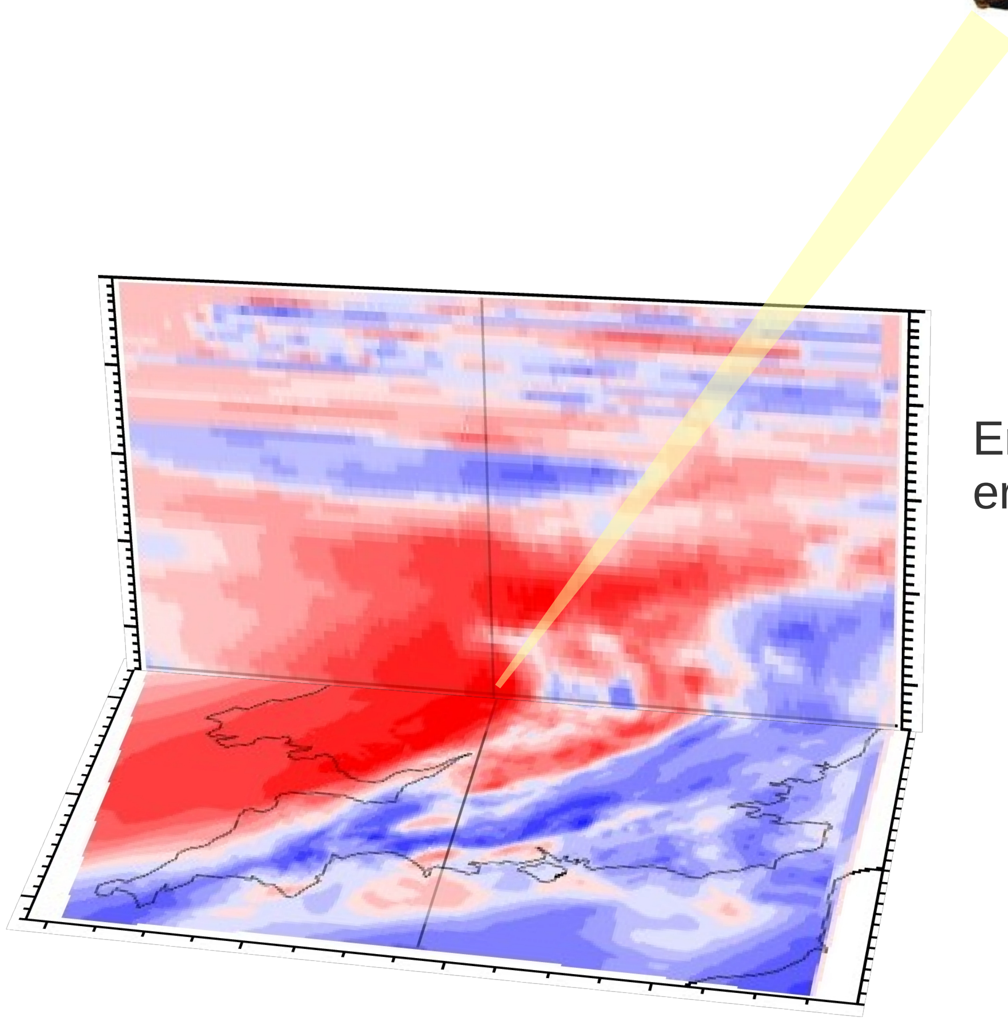
Forecast error covariance statistics can be approximated by an *N*-member ensemble for ensemble Kalman filtering:

$$\mathbf{P}^{\mathrm{f}} \rightarrow \mathbf{P}^{\mathrm{f}}_{N} = \frac{1}{N-1}\sum_{l=1}^{N}(\mathbf{x}_l - \bar{\mathbf{x}})(\mathbf{x}_l - \bar{\mathbf{x}})^{\mathrm{T}}$$

☑ Easy to implement ☑ Flow/weather dependent ☒ Full rank estimate of $\mathbf{P}^{\mathrm{f}}$

University of Reading National Centre for Earth Observation NATURAL ENVIRONMENT RESEARCH COUNCIL esa DIAMET

Consequences of sampling error ...

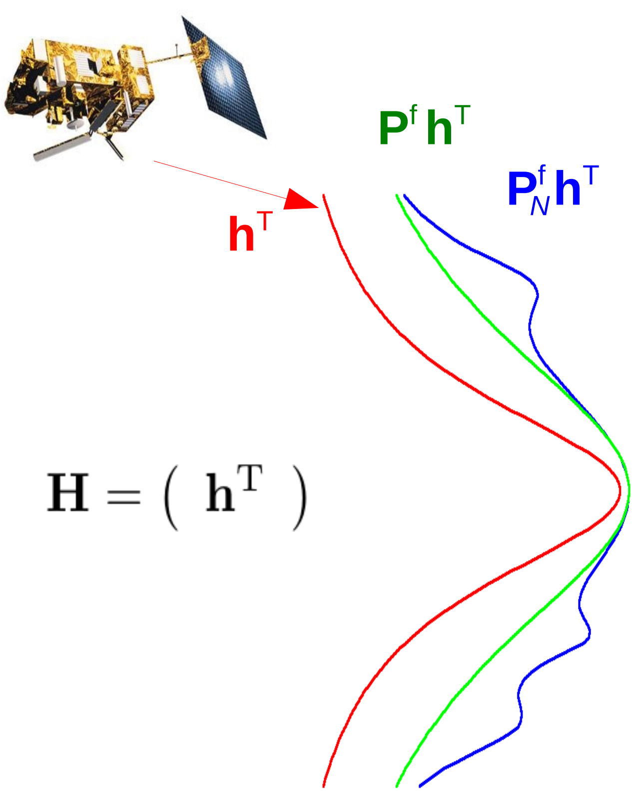


Ensemble-derived forecast error correlation function

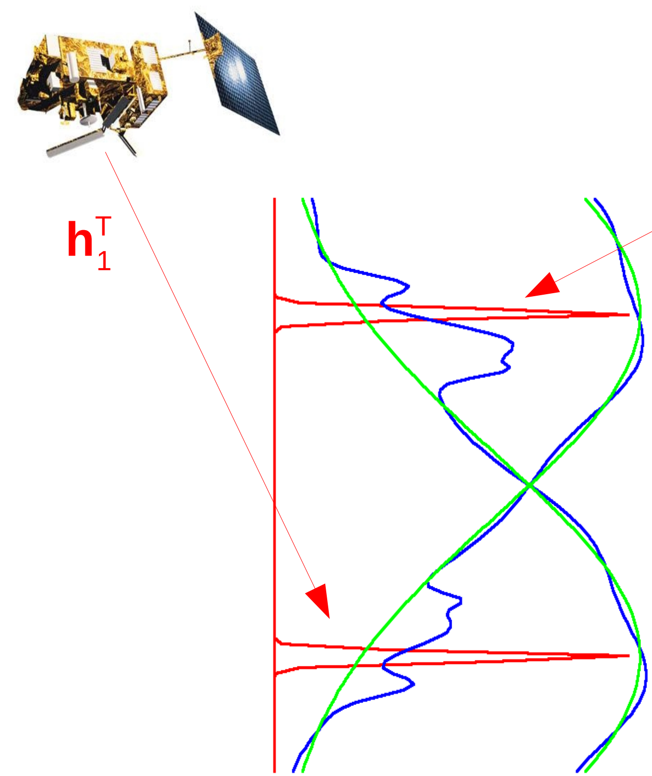
... consequences of sampling error

$$\mathbf{x}^{\mathrm{a}} = \mathbf{x}^{\mathrm{b}} + \mathbf{P}^{\mathrm{f}}\mathbf{H}^{\mathrm{T}}\left(\mathbf{R} + \mathbf{H}\mathbf{P}^{\mathrm{f}}\mathbf{H}^{\mathrm{T}}\right)^{-1}\left(\mathbf{y} - \mathbf{H}\mathbf{x}^{\mathrm{b}}\right)$$

ONE ob with a BROAD Jacobian	TWO obs each with NARROW Jacobians



$$\mathbf{H} = \left(\ \mathbf{h}^{\mathrm{T}}\ \right)$$



$$\mathbf{H} = \begin{pmatrix} \mathbf{h}_1^{\mathrm{T}} \\ \mathbf{h}_2^{\mathrm{T}} \end{pmatrix}$$

$$\left(\mathbf{H}\mathbf{P}^{\mathrm{f}}\mathbf{H}^{\mathrm{T}}\right)_{11} = \mathbf{h}^{\mathrm{T}}\mathbf{P}^{\mathrm{f}}_{N}\mathbf{h}$$

Sampling error will look like an extra obs err variance

$$\text{E.g.}\quad \left(\mathbf{H}\mathbf{P}^{\mathrm{f}}\mathbf{H}^{\mathrm{T}}\right)_{12} = \mathbf{h}_1^{\mathrm{T}}\mathbf{P}^{\mathrm{f}}_{N}\mathbf{h}_2$$

Sampling error will look like an extra obs err covariance

Localization as a means of mitigating sampling error

- The error in the sample correlation between two variables *x* and *y* is:

$$\text{Error in COR}(x,y) \sim \frac{1-\text{COR}^2(x,y)}{\sqrt{N-1}}$$

- Expect largest sampling error when actual correlation is small.
- Expect actual correlation to be small when separation between *x* and *y* is large.
- Introduce separation-dependent damping of the correlations and covariances.
  - → Localization

$$\mathbf{P}_N^{\mathrm{f}} \rightarrow \overset{\circ}{\mathbf{P}}{}_N^{\mathrm{f}} = \mathbf{P}_N^{\mathrm{f}} \circ \mathbf{\Omega}$$

Localization

$$\mathbf{P}^{\mathrm{f}}_{N} \rightarrow \overset{\circ}{\mathbf{P}}{}^{\mathrm{f}}_{N} = \mathbf{P}^{\mathrm{f}}_{N} \circ \mathbf{\Omega}$$

- Most applications choose Ω to be a decreasing function of separation.
- BUT
  - Some covariances do have regions that should increase with distance.
  - Localization can remove long-range correlations, but they can destroy other important properties (e.g. balance).
  - Balance is thought to be the reason why modern data assimilation for weather forecasting is so successful, so DON'T WANT TO DESTROY THIS!

Wish list

$$\mathbf{P}^{\mathrm{f}}_{N} \rightarrow \overset{\circ}{\mathbf{P}}{}^{\mathrm{f}}_{N} = \mathbf{P}^{\mathrm{f}}_{N} \circ \boldsymbol{\Omega}$$

- **Use a finite ensemble where:**
  - Unphysical correlations are removed.
  - Preserves physical properties like 'balance'.

- **Examine five ways of modelling Ω:**
  - Spectral method (static Ω).
    - Types I and II.
  - SENCORP-like method (adaptive Ω – Bishop et al.).
  - ECORAP method (adaptive Ω – Bishop et al.).
    - Types I and II.

- **Remainder of this talk:**
  - Describe these methods.
  - Explain the balance diagnostics.
  - Extracting balance information from the covariances.
  - Implied correlation functions and balance properties of the methods.

The localization methods

Remember ...

$$\left.\begin{aligned}\mathbf{P}_N^{\mathrm{f}\,n\times n} &= \frac{1}{N-1}\mathbf{X}^{n\times N}\mathbf{X}^{n\times N\,\mathrm{T}} \\ \Omega^{n\times n} &= \frac{1}{K-1}\mathbf{K}^{n\times K}\mathbf{K}^{n\times K\,\mathrm{T}}\end{aligned}\right\} \quad \overset{\circ}{\mathbf{P}}{}_N^{\mathrm{f}} = \mathbf{P}_N^{\mathrm{f}} \circ \mathbf{\Omega}$$

1. Spectral I (static, univariate *K* = No. of waves ✕ No. of parameters)

$$\Omega^{n\times n}_{\mathrm{spec(I)}} \propto \begin{pmatrix} \mathbf{F}\Lambda_{p_1}\mathbf{F}^{\mathrm{T}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{F}\Lambda_{p_2}\mathbf{F}^{\mathrm{T}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{F}\Lambda_{p_3}\mathbf{F}^{\mathrm{T}} \end{pmatrix} \quad \mathbf{K}^{n\times K}_{\mathrm{spec}(I)} \propto \begin{pmatrix} \mathbf{F}\Lambda_{p_1}^{1/2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{F}\Lambda_{p_2}^{1/2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{F}\Lambda_{p_3}^{1/2} \end{pmatrix}$$

2. Spectral II (static multivariate, *K* = No. of waves)

$$\Omega^{n\times n}_{\mathrm{spec(II)}} \propto \begin{pmatrix} \mathbf{F}\Lambda_{p_1}\mathbf{F}^{\mathrm{T}} & \mathbf{F}\Lambda_{p_1}^{1/2}\Lambda_{p_2}^{1/2}\mathbf{F}^{\mathrm{T}} & \mathbf{F}\Lambda_{p_1}^{1/2}\Lambda_{p_3}^{1/2}\mathbf{F}^{\mathrm{T}} \\ \mathbf{F}\Lambda_{p_2}^{1/2}\Lambda_{p_1}^{1/2}\mathbf{F}^{\mathrm{T}} & \mathbf{F}\Lambda_{p_2}\mathbf{F}^{\mathrm{T}} & \mathbf{F}\Lambda_{p_2}^{1/2}\Lambda_{p_3}^{1/2}\mathbf{F}^{\mathrm{T}} \\ \mathbf{F}\Lambda_{p_3}^{1/2}\Lambda_{p_1}^{1/2}\mathbf{F}^{\mathrm{T}} & \mathbf{F}\Lambda_{p_3}^{1/2}\Lambda_{p_2}^{1/2}\mathbf{F}^{\mathrm{T}} & \mathbf{F}\Lambda_{p_3}\mathbf{F}^{\mathrm{T}} \end{pmatrix} \quad \mathbf{K}^{n\times K}_{\mathrm{spec}(II)} \propto \begin{pmatrix} \mathbf{F}\Lambda_{p_1}^{1/2} \\ \mathbf{F}\Lambda_{p_2}^{1/2} \\ \mathbf{F}\Lambda_{p_3}^{1/2} \end{pmatrix}$$

n: size of state   *N*: No. of actual members   *K*: No. of columns in **K** (sqrt of **Ω**)

3. SENCORP (Bishop et al., *K* = *N*^{Q})

$$\Omega^{n\times n}_{\mathrm{SENCORP}} \propto \mathbf{C}^{\circ Q} = \mathbf{C}\circ\mathbf{C}\circ\cdots\circ\mathbf{C}$$

$$\mathbf{K}^{n\times K}_{\mathrm{SENCORP}} \propto \left(\begin{array}{c}\text{every possible}\\ \text{combination of}\\ Q\text{ smoothed members}\end{array}\right)$$

$$\mathbf{C}^{n\times n} = \frac{1}{L}\mathbf{W}^{n\times N}\mathbf{W}^{n\times N\,\mathrm{T}}$$

4. ECORAP I (Bishop et al., *K* = No. of waves × No. of parameters)

$$\Omega^{n\times n}_{\mathrm{ECORAP(I)}} \propto \mathbf{C}^{\circ Q}\begin{pmatrix}\mathbf{F}\Lambda_{p_1}\mathbf{F}^{\mathrm{T}} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{F}\Lambda_{p_2}\mathbf{F}^{\mathrm{T}} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{F}\Lambda_{p_3}\mathbf{F}^{\mathrm{T}}\end{pmatrix}\mathbf{C}^{\circ Q}$$

$$\mathbf{K}^{n\times K}_{\mathrm{ECORAP}(I)} \propto \mathbf{C}^{\circ Q}\begin{pmatrix}\mathbf{F}\Lambda^{1/2}_{p_1} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{F}\Lambda^{1/2}_{p_2} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{F}\Lambda^{1/2}_{p_3}\end{pmatrix}$$

5. ECORAP II (Bishop et al., *K* = No. of waves)

$$\Omega^{n\times n}_{\mathrm{ECORAP(II)}} \propto \mathbf{C}^{\circ Q}\begin{pmatrix}\mathbf{F}\Lambda_{p_1}\mathbf{F}^{\mathrm{T}} & \mathbf{F}\Lambda^{1/2}_{p_1}\Lambda^{1/2}_{p_2}\mathbf{F}^{\mathrm{T}} & \mathbf{F}\Lambda^{1/2}_{p_1}\Lambda^{1/2}_{p_3}\mathbf{F}^{\mathrm{T}}\\ \mathbf{F}\Lambda^{1/2}_{p_2}\Lambda^{1/2}_{p_1}\mathbf{F}^{\mathrm{T}} & \mathbf{F}\Lambda_{p_2}\mathbf{F}^{\mathrm{T}} & \mathbf{F}\Lambda^{1/2}_{p_2}\Lambda^{1/2}_{p_3}\mathbf{F}^{\mathrm{T}}\\ \mathbf{F}\Lambda^{1/2}_{p_3}\Lambda^{1/2}_{p_1}\mathbf{F}^{\mathrm{T}} & \mathbf{F}\Lambda^{1/2}_{p_3}\Lambda^{1/2}_{p_2}\mathbf{F}^{\mathrm{T}} & \mathbf{F}\Lambda_{p_3}\mathbf{F}^{\mathrm{T}}\end{pmatrix}\mathbf{C}^{\circ Q}$$

$$\mathbf{K}^{n\times K}_{\mathrm{spec}(II)} \propto \mathbf{C}^{\circ Q}\begin{pmatrix}\mathbf{F}\Lambda^{1/2}_{p_1}\\ \mathbf{F}\Lambda^{1/2}_{p_2}\\ \mathbf{F}\Lambda^{1/2}_{p_3}\end{pmatrix}$$

Balance diagnostics

$$t_A + t_B = 0$$

$$\rho_{AB} = \langle (t_A - \langle t_A \rangle)(t_B - \langle t_B \rangle) \rangle$$

$$\text{Balanced if } \rho_{AB} \sim -1$$

Examples:

	t_A	t_B
Geostrophic balance	Mass terms	Wind terms
Hydrostatic balance	Pressure term	Density term

Extraction of balance information

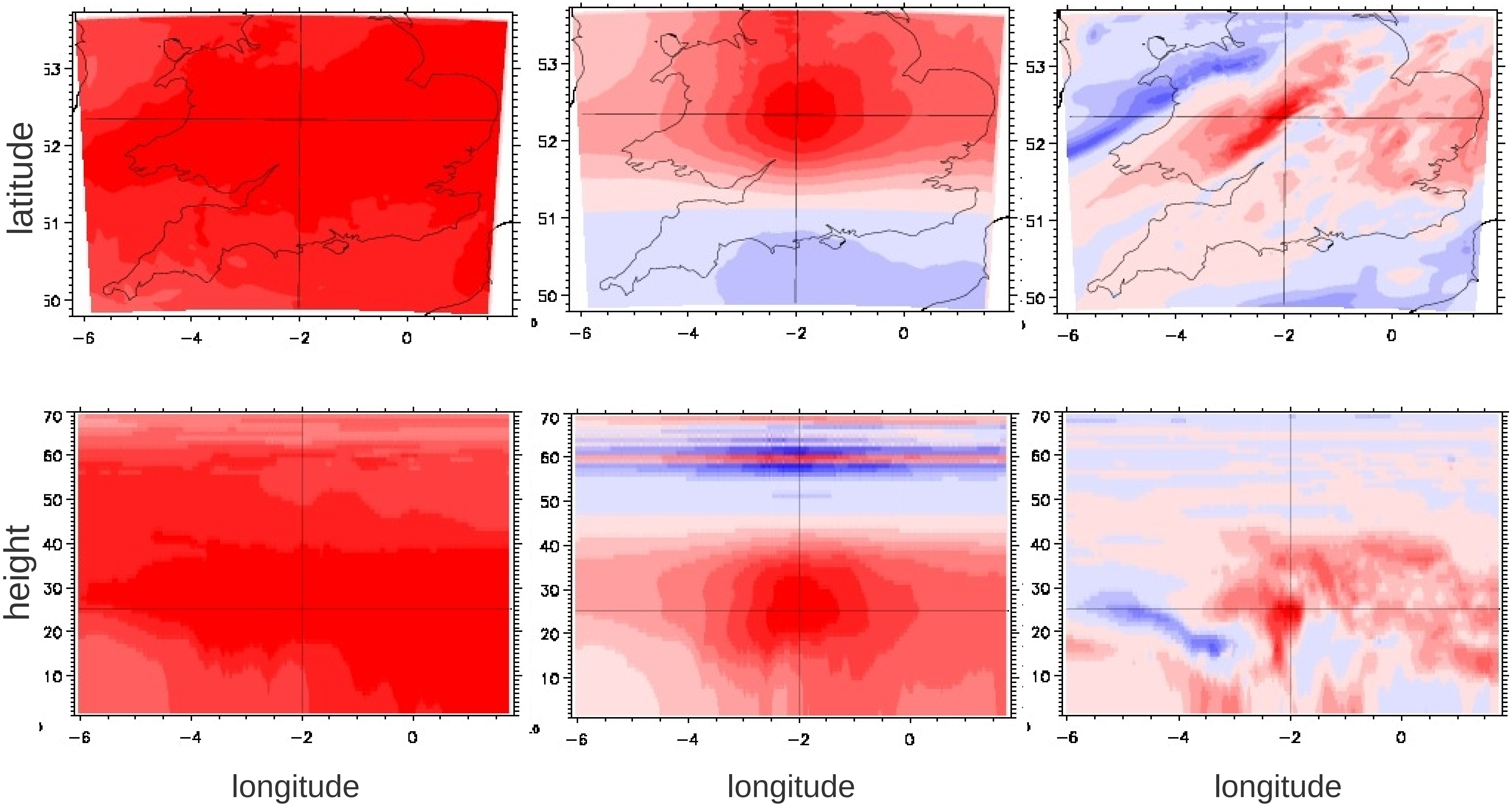
$$\begin{aligned}
\mathbf{P}_N^{\mathrm{f}} &= \frac{1}{N-1}\mathbf{X}\mathbf{X}^{\mathrm{T}} \\
\Omega &= \frac{1}{K-1}\mathbf{K}\mathbf{K}^{\mathrm{T}} \\
\mathring{\mathbf{P}}_N^{\mathrm{f}} &= \mathbf{P}_N^{\mathrm{f}} \circ \boldsymbol{\Omega}
\end{aligned}$$

$$\begin{aligned}
\left(\mathring{\mathbf{P}}_N^{\mathrm{f}}\right)_{ij} &= \left(\mathbf{P}_N^{\mathrm{f}}\right)_{ij} \circ \boldsymbol{\Omega}_{ij} \\
&= \frac{1}{(N-1)(K-1)} \sum_{l,k} \mathbf{X}_{il}\mathbf{X}_{jl}\mathbf{K}_{ik}\mathbf{K}_{jk} \\
&= \frac{1}{(N-1)(K-1)} \sum_{l,k} \left(\mathbf{X}_{il}\mathbf{K}_{ik}\right)\left(\mathbf{X}_{jl}\mathbf{K}_{jk}\right)
\end{aligned}$$

Effective ensemble of *NK* members

Example localized correlation functions (provisional)

No localization (pressure) **1, 2. Static (types I and II)** **3. SENCORP**



Balance diagnostics (provisional)

No localization		Spectral (I or II?)		SENCORP		ECORAP (I or II?)	
Geo	Hydro	Geo	Hydro	Geo	Hydro	Geo	Hydro