THE DOUBLE PENDULUM (MODIFIED)

A different kind of double pendulum consists of an extra arm of length l_1 (Fig. 2). This system has a slightly different Lagrangian from that of the usual double pendulum, but still has the same two degrees of freedom and so the same method of solution can be used. Expressions for the elements of the matrices **A** and **B**, and components of the vector ***g*** should be substituted by those given below.

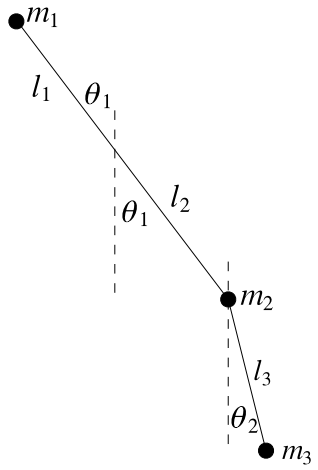

Figure 2: The modified double pendulum set-up. The lengths l_1, l_2 and l_3, and the masses m_1, m_2 and m_3 are fixed. The system has the degrees of freedom θ_1 and θ_2.

7. CONSTRUCT THE MODIFIED LAGRANGIAN

The components of the Lagrangian for this modified system are,

$$V = gm_1l_1\cos\theta_1 - gm_2l_2\cos\theta_1 - gm_3\left(l_2\cos\theta_1 + l_3\cos\theta_2\right), \tag{7.1}$$

$$T = \tfrac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \tfrac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2) + \tfrac{1}{2}m_3(\dot{x}_3^2 + \dot{y}_3^2). \tag{7.2}$$

The relevant transformations are,

$$x_1 = -l_1\sin\theta_1, \qquad \dot{x}_1 = -l_1\cos\theta_1\dot{\theta}_1, \tag{7.3}$$

$$y_1 = l_1\cos\theta_1, \qquad \dot{y}_1 = -l_1\sin\theta_1\dot{\theta}_1, \tag{7.4}$$

$$x_2 = l_2\sin\theta_1 \qquad \dot{x}_1 = l_2\cos\theta_1\dot{\theta}_1, \tag{7.5}$$

$$y_2 = -l_2\cos\theta_1 \qquad \dot{y}_2 = l_2\sin\theta_1\dot{\theta}_1, \tag{7.6}$$

$$x_3 = l_2\sin\theta_1 + l_3\sin\theta_2, \qquad \dot{x}_2 = l_2\cos\theta_1\dot{\theta}_1 + l_3\cos\theta_2\dot{\theta}_2, \tag{7.7}$$

$$y_3 = -l_2\cos\theta_1 - l_3\cos\theta_2, \qquad \dot{y}_2 = l_2\sin\theta_1\dot{\theta}_1 + l_3\sin\theta_2\dot{\theta}_2. \tag{7.8}$$

Inserting Eqs. (7.3) to (7.8) into Eq. (7.2) (and simplifying), the Lagrangian becomes,

$$L = \tfrac{1}{2}m_1l_1^2\dot{\theta}_1^2 + \tfrac{1}{2}m_2l_2^2\dot{\theta}_1^2 + \tfrac{1}{2}m_3\left(l_2^2\dot{\theta}_1^2 + l_3^2\dot{\theta}_2^2 + 2l_2l_3\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)\right)$$
$$- gm_1l_1\cos\theta_1 + gm_2l_2\cos\theta_1 + gm_3\left(l_2\cos\theta_1 + l_3\cos\theta_2\right). \tag{7.9}$$

8. MATRIX AND VECTOR ELEMENTS FOR THE MODIFIED SYSTEM

The following, for the modified system, replace Eqs. (5.1) to (5.10),

$$\frac{\partial^2 L}{\partial \dot{\theta}_1 \partial \theta_1} = -m_3 l_2 l_3 \dot{\theta}_2 \sin(\theta_1 - \theta_2), \tag{7.1}$$

$$\frac{\partial^2 L}{\partial \dot{\theta}_1 \partial \theta_2} = m_3 l_2 l_3 \dot{\theta}_2 \sin(\theta_1 - \theta_2), \tag{7.2}$$

$$\frac{\partial^2 L}{\partial \dot{\theta}_2 \partial \theta_1} = -m_3 l_2 l_3 \dot{\theta}_1 \sin(\theta_1 - \theta_2), \tag{7.3}$$

$$\frac{\partial^2 L}{\partial \dot{\theta}_2 \partial \theta_2} = m_3 l_2 l_3 \dot{\theta}_1 \sin(\theta_1 - \theta_2), \tag{7.4}$$

$$\frac{\partial^2 L}{\partial \dot{\theta}_1^2} = m_1 l_1^2 + (m_2 + m_3) l_2^2, \tag{7.5}$$

$$\frac{\partial^2 L}{\partial \dot{\theta}_1 \partial \dot{\theta}_2} = m_3 l_2 l_3 \cos(\theta_1 - \theta_2), \tag{7.6}$$

$$\frac{\partial^2 L}{\partial \dot{\theta}_2 \partial \dot{\theta}_1} = m_3 l_2 l_3 \cos(\theta_1 - \theta_2), \tag{7.7}$$

$$\frac{\partial^2 L}{\partial \dot{\theta}_2^2} = m_3 l_3^2, \tag{7.8}$$

$$\frac{\partial L}{\partial \theta_1} = -m_3 l_2 l_3 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + g m_1 l_1 \sin\theta_1 - g(m_2 + m_3) l_2 \sin\theta_1, \tag{7.9}$$

$$\frac{\partial L}{\partial \theta_2} = m_3 l_2 l_3 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - g m_3 l_3 \sin\theta_2. \tag{7.10}$$